Homework 5

1. Let \( \psi \) be given by the wavelet equation:
\[
\psi(t) = \sqrt{2} \sum_k g_k \phi(2t - k).
\]
Show that \( \{\psi(t - k)\}_k \) is an ON set if and only if
\[
|G(\gamma)|^2 + |G(\gamma + 1/2)|^2 = 2,
\]
where \( G(\gamma) \) is the Fourier Transform of the sequence \( \{g_k\} \).

2. Let \( \{V_j, \phi\} \) be a given MRA. Let \( \phi \in V_0 \) and \( \psi \in W_0 \) be given by the scaling and wavelet equations, respectively,
\[
\phi = \sum_k h_k \phi_k,
\]
and
\[
\psi = \sum_k g_k \phi_k.
\]
Suppose that
\[
\phi_k(t) = \sum_n a_{n,k} \phi(t - n) + \sum_n b_{n,k} \psi(t - n).
\]
Show that
\[
a_{n,k} = \overline{h_{-2n}},
\]
and
\[
b_{n,k} = \overline{g_{-2n}}.
\]

3. Let \( f \in L^2(\mathbb{R}) \). Assume \( f = \sum_j f_j \) where \( f_i \perp f_j \) for all \( i \neq j \). Show that \( \|f\|^2 = \sum_j \|f_j\|^2 \).

4. Let \( \{V_j, \phi\} \) be a given MRA. Show that
\[
\phi_k(t) = \sum_n h_{k-2n} \phi(t - n) + \sum_n g_{k-2n} \psi(t - n)
\]
if and only if
\[
\frac{|H(\gamma)|^2 + |G(\gamma)|^2}{H(\gamma)\overline{G(\gamma)} + H(\gamma + \frac{1}{2})\overline{G(\gamma + \frac{1}{2})}} = 0.
\]

Hint: You would need to take the Fourier Transform of (1). You will see that there is a need to separate the equation into two by considering even \( k \) and odd \( k \). You will also need to use
\[
\Phi(\gamma) \equiv \sum_k |\phi(\gamma + k)|^2 = 1.
\]