Homework 3

1. Let $U$ be the coefficient map defined by a Bessel sequence $\{x_n\} \subseteq \mathcal{H}$. Show that the null space (kernel) of $U$, $\mathcal{N}(U) = \text{span}\{x_n\}^\perp$.

2. Let $\mathcal{P}_{X,Y}$ be a nonorthogonal projection onto $X$ along a complementary subspace $Y$ (of $X$) in $\mathcal{H}$. Show that $\mathcal{P}_{X,Y}^* = \mathcal{P}_{Y^\perp,X^\perp}$, i.e., $\mathcal{P}_{X,Y}$ is also a projection.

3. Let $\mathcal{X} \equiv \text{span}\{u_1, u_2\} \subset \mathbb{C}^3$, where $u_1 = <1,0,1>$ and $u_2 = <1,0,0>$. Find three non-colinear vectors $\{x_n^1, x_n^2, x_n^3\}$ in $\mathcal{X}$. Find then a parametric formula for a set of PFFS-duals $\{x_n\}_{n=1}^\infty$.

4. (Optional) Can you find the most general formula for all PFFS-duals of the above problem.

5. Problem 1 is a PFFS example where $\{x_n^*\}$ is in $\mathcal{X}$, but $\{x_n\}$ is not. We could find an example where even $\{x_n^*\}$ is not in a subspace $\mathcal{X}$.

   Let $\mathcal{X} = \text{span}\{u_1\} \subset \mathbb{C}^3$ where $u_1$ is a line with the direction vector $<1,-1,1>$ and passes through the point $(1,2,1)$.

   (a) Find a plane $Q$ containing $\mathcal{X}$, and then select 3 vectors $\{x_n^*\}_{n=1}^3$ in plane $Q$.

   (b) Find the orthogonal complement of $\mathcal{X}$ in $\mathbb{C}^3$. [Hint: it should be a plane].

   (c) Find a parametric formula for PFFS-duals of $\{x_n^*\}$. [Hint: using the relationship theorem between PFFS for $\mathcal{X}$ and a frame of $\mathcal{X}$. Find a frame of $\mathcal{X}$ first, then find its dual, etc.]