Homework 2

1. Assume that a Gaussian (window) function achieves the bound of the uncertainty principle. Explain why one can not form a Gabor basis with Gaussian windows.

2. Let \( \{x_n\} \) be a frame of \( \mathcal{H} \), and let \( S \) be the corresponding frame operator. Prove that

\[
\forall f \in \mathcal{H}, \quad \sum_n \|\langle f, S^{-1} x_n \rangle\|^2 \leq \sum_n |c(n)|^2,
\]

where \( \{c(n)\} \) are all possible coefficients such that \( f = \sum_n c_n x_n \) for a given \( f \in \mathcal{H} \). What does this mean?

3. Let \( \{x_n\} \) be a frame for \( \mathcal{H} \) with the coefficient mapping \( U : \mathcal{H} \to l^2 \). Suppose that \( V^* \) is a bounded left inverse of \( U \). Show that \( \{x_n^* \equiv V^* e_n\} \) is a dual frame of \( \{x_n\} \).

4. Let \( S \) be a frame operator defined by a Gabor frame \( \{g_{m,n}\} \). Let \( T_n \) and \( E_m \) be the translation and modulation operators defined by \( T_n f(t) = f(t-nT) \) and \( E_m f = f(t)e^{2\pi i mt/N} \), respectively. Verify that

\[
S(E_m T_n h) = E_m T_n (Sh), \quad \forall h \in \mathcal{H},
\]

and deduce that \( S^{-1} (E_m T_n g) = E_m T_n (S^{-1} g) \). What does this last equation imply?

5. (Optional) Can you think of an example (about the application of frames) where the use of alternative (non-standard) dual frames is demanded?

6. (Optional) A transversal filter can in fact be formulated in the frame context. Can you provide a frame theoretical formulation of a transversal filter?