A MAXIMUM PRINCIPLE FOR THE FOCK SPACE

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Abstract. It is shown that the Fock space of entire functions has a maximum principle analogous to that in the Bergman space.

§1. Introduction.
The Fock space $F$ is the set of entire functions with

$$
\|f\| = \left\{ \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, dA(z) \right\}^{1/2} < \infty,
$$

where $\mathbb{C}$ is the complex plane and $dA$ denotes Lebesgue area measure. In this note we prove the following result.

Theorem. There is a positive constant $c$ with the property that whenever $f$ and $g$ are entire functions satisfying $|f(z)| \leq |g(z)|$ for $|z| > c$, then $\|f\| \leq \|g\|$.

This theorem has its roots in the so-called Korenblum Maximum Principle, which deals with the Bergman space of square-integrable analytic functions on the unit disk $\mathbb{D}$. It states that there is a constant $a \in (0, 1)$ such that whenever $f$ and $g$ are functions analytic in $\mathbb{D}$ with $|f(z)| \leq |g(z)|$ for $a < |z| < 1$, then the $L^2$ norm of $f$ is bounded above by the $L^2$ norm of $g$. Conjectured by Korenblum [2], the result was proved by Hayman [3]. Hinkkanen [4] later showed that the principle holds more generally for the $L^p$ Bergman space, where $1 \leq p < \infty$. Moreover, he improved upon the constant obtained by Hayman. To our knowledge, the best estimate on the optimal constant is that it lies in the interval $(0.157... , 0.69472...)$, where the lower bound was obtained by Hinkkanen and the upper by Wang [5]. See [1] for more references on this problem.

In this note we show that the general scheme of Hinkkanen may be applied to our situation. In fact, parts of the proof are simplified, because the geometry of the region involved is easier to handle. As in the Bergman space, it is of some interest to find the optimal constant for which the theorem holds, but we will not be concerned with that here. We do make the remark that the optimal constant is at most 1. This can be seen by setting $f(z) = c$ and $g(z) = z$. 

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The proof can be extended without much effort to the generalized Fock space, which consists of entire functions satisfying
\[
\int_{\mathbb{C}} |f(z)|^p e^{-\varphi(z)} \, dA(z) < \infty, \tag{1}
\]
where \(1 \leq p < \infty\) and \(\varphi\) is a continuous real-valued function with \(e^{-\varphi} \in L^1\). For simplicity of exposition we deal here only with the case \(p = 2\) and \(\varphi(z) = |z|^2\).

We note also that because of Hartogs' theorem it is only in one dimension that this result is interesting.

\section{The proof.}

Let \(c \in (0,1)\) be a constant to be determined later, and suppose that \(f\) and \(g\) are entire functions satisfying \(|f(z)| \leq |g(z)|\) for \(z \in A(c)\), where \(A(c) = \{z \in \mathbb{C} : |z| > c\}\). Our goal is to show that
\[
\int_{D(c)} (|f(z)|^2 - |g(z)|^2) e^{-|z|^2} \, dA(z) \leq \int_{A(c)} (|g(z)|^2 - |f(z)|^2) e^{-|z|^2} \, dA(z),
\]
where \(D(c)\) is the open disk of radius \(c\) centered at the origin.

Define the function \(\omega = f/g\), which by hypothesis is analytic and satisfies \(|\omega| \leq 1\) in \(A(c)\). In fact, we may assume, without loss of generality, that \(|\omega| < 1\) in \(A(c)\), since otherwise \(|f| = |g|\) in \(\mathbb{C}\) and the result holds trivially. We may likewise assume that \(f\) is not identically equal to \(0\).

For \(\rho \geq 1\), choose \(\zeta_\rho\) such that \(|\zeta_\rho| = \rho\) and \(|\omega(\zeta_\rho)| = \sup\{|\omega(z)| : |z| = \rho\}\), and define \(\omega_\rho = \omega(\zeta_\rho)\).

Let \(0 < r < c < 1 \leq \rho\). Following Hinkkanen, we use the inequality \(|\alpha|^2 - |\beta|^2 \leq 2|\alpha^2 - \alpha|\beta|\), which holds for all complex numbers \(\alpha\) and \(\beta\), to obtain
\[
\int_0^{2\pi} (|f(re^{i\theta})|^2 - |g(re^{i\theta})|^2) \, d\theta \leq \int_0^{2\pi} (|f(re^{i\theta})|^2 - |\omega_\rho g(re^{i\theta})|^2) \, d\theta
\leq 2 \int_0^{2\pi} |f^2(re^{i\theta}) - \omega_\rho f(re^{i\theta}) g(re^{i\theta})| \, d\theta
\leq 2 \int_0^{2\pi} |f^2(\rho e^{i\theta}) - \omega_\rho f(\rho e^{i\theta}) g(\rho e^{i\theta})| \, d\theta
= 2 \int_0^{2\pi} |\omega(z)| \frac{|\omega(z) - \omega_\rho|}{1 - |\omega(z)|^2} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) \, d\theta
\leq 2 \gamma(\rho) \int_0^{2\pi} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) \, d\theta.
\]

where
\[
\gamma(\rho) = \sup \left\{ \frac{|\omega(z) - \omega_\rho|}{1 - |\omega(z)|^2} : |z| = \rho \right\}.
\]

Multiplying both sides of this inequality by \(r^{-r^2}\) and then integrating from 0 to \(c\) with respect to \(r\) yields
\[
\int_{D(c)} (|f(z)|^2 - |g(z)|^2) e^{-|z|^2} \, dA(z) \leq \gamma(\rho)(1 - e^{-c^2}) \int_0^{2\pi} (|g(\rho e^{i\theta})|^2 - |f(\rho e^{i\theta})|^2) \, d\theta.
\]
Define $\gamma = \sup_{\rho \geq 1} \gamma(\rho)$. Multiply both sides of this last inequality by $\rho e^{-\rho^2}$ and integrate from $1$ to $\infty$ with respect to $\rho$ to arrive at
\[
\int_{D(c)} (|f(z)|^2 - |g(z)|^2)e^{-|z|^2} \, dA(z) \leq 2\gamma e(1 - e^{-c^2}) \int_{A(1)} (|g(z)|^2 - |f(z)|^2)e^{-|z|^2} \, dA(z) \leq 2\gamma e(1 - e^{-c^2}) \int_{A(c)} (|g(z)|^2 - |f(z)|^2)e^{-|z|^2} \, dA(z).
\]

It remains to show that $2\gamma e(1 - e^{-c^2}) \leq 1$ for sufficiently small $c$. (Note that $\gamma$ is really a function of $c$.) To this end, we recall that the pseudohyperbolic distance $d$ between two points $\alpha, \beta \in \mathbb{D}$ is given by the formula
\[
d(\alpha, \beta) = \frac{|\alpha - \beta|}{1 - |\alpha||\beta|}.
\]
Define the function $H : \mathbb{D}^* \rightarrow \mathbb{D}$, where $\mathbb{D}^*$ is the punctured unit disk, by the equation $H(\eta) = \omega(c/\eta)$. This is analytic in $\mathbb{D}^*$, and since $\omega$ is bounded in $A(c)$, $H$ can be extended to be analytic on all of $\mathbb{D}$. By the Schwarz-Pick lemma, we then have for $|z| = \rho$,
\[
d(\omega(z), \omega_\rho) \leq d\left(\frac{c}{z}, \frac{c}{\zeta_\rho}\right) \leq \frac{2c\rho}{\rho^2 + c^2} \leq \frac{2c}{1 + c^2}.
\]
We use the identity
\[
\frac{|\alpha - \beta|}{1 - |\alpha|^2} = \frac{d(\alpha, \beta)}{\sqrt{1 - |\beta|^2}} \frac{1 - |\beta|^2}{\sqrt{1 - |\alpha|^2}}
\]
to show that for $|z| = \rho$, we have
\[
\gamma \leq \frac{\omega(z) - \omega_\rho}{1 - |\omega(z)|^2} \leq \frac{2c(1 + c^2)^{-1}}{\sqrt{1 - (2c(1 + c^2)^{-1})^2}} = \frac{2c}{1 - c^2}.
\]
Therefore, we have
\[
\int_{D(c)} (|f(z)|^2 - |g(z)|^2)e^{-|z|^2} \, dA(z) \leq 4c(1 - e^{-c^2}) \int_{A(c)} (|g(z)|^2 - |f(z)|^2)e^{-|z|^2} \, dA(z).
\]
By choosing $c$ so close to zero that the expression in front of the integral is bounded above by $1$, we have proved (1) and therefore completed the proof of the theorem.

References

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