29. A company manufactures only one product. The quantity, \( q \), of this product produced per month depends on the amount of capital, \( K \), invested (i.e., the number of machines the company owns, the size of its building, and so on) and the amount of labor, \( L \), available each month. We assume that \( q \) can be expressed as a Cobb-Douglas production function:

\[
q = cK^{\alpha}L^{\beta}
\]

where \( c, \alpha, \beta \) are positive constants, with \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). In this problem we will see how the Russian government could use a Cobb-Douglas function to estimate how many people a newly privatized industry might employ. A company in such an industry has only a small amount of capital available to it and needs to use all of it, so \( K \) is fixed. Suppose \( L \) is measured in man-hours per month, and that each man-hour costs the company \( w \) rubles (a ruble is the unit of Russian currency). Suppose the company has no other costs besides labor, and that each unit of the good can be sold for a fixed price of \( p \) rubles. How many man-hours of labor per month should the company use in order to maximize its profit?

4.5 AVERAGE COST

To maximize profit, a company arranges production to equalize marginal cost and marginal revenue. But how do we know if the company makes money? It turns out that whether the maximum profit is positive or negative is determined by the company’s average cost of production. Average cost also tells us about the behavior of similar companies in an industry. If average costs are low, more companies will enter the market; if average costs are high, companies will leave the market.

In this section, we see how average cost can be calculated and visualized, and the relationship between average and marginal cost.

What Is Average Cost?

The average cost is the cost per unit of producing a certain quantity; it is the total cost divided by the number of units produced.

If the cost of producing a quantity \( q \) is \( C(q) \), then the average cost, \( a(q) \), of producing a quantity \( q \) is given by

\[
a(q) = \frac{C(q)}{q}.
\]

Although both are measured in the same units, for example, dollars per item, be careful not to confuse the average cost with the marginal cost (the cost of producing the next item).

Example 1

A salsa company has cost function \( C(q) = 0.01q^3 - 0.6q^2 + 13q + 1000 \) (in dollars), where \( q \) is the number of cases of salsa produced. If 100 cases are produced, find the average cost per case.

Solution

The total cost of producing the 100 cases is given by

\[
C(100) = 0.01(100^3) - 0.6(100^2) + 13(100) + 1000 = \$6300.
\]
(d) See Figure 4.62. Marginal cost is the derivative of $C(q) = q^2 - 6q^2 + 15q$,
$$MC(q) = 3q^2 - 12q + 15.$$ 

(e) At $q = 3$, we have
$$\text{Marginal cost} = 3 \cdot 3^2 - 12 \cdot 3 + 15 = 6.$$ 
$$\text{Average cost} = 3^2 - 6 \cdot 3 + 15 = 6.$$ 
Thus, marginal and average cost are equal at $q = 3$. This result can be seen in Figure 4.62 since the marginal cost curve cuts the average cost curve at the minimum average cost.

**Problems for Section 4.5**

1. Figure 4.63 shows cost with $q = 10,000$ marked.
   (a) Find the average cost when the production level is 10,000 units and interpret it.
   (b) Represent your answer to part (a) graphically.
   (c) At approximately what production level is average cost minimized?

2. The cost of producing $q$ items is $C(q) = 2500 + 12q$ dollars.
   (a) What is the marginal cost of producing the 100th item? the 1000th item?
   (b) What is the average cost of producing 100 items? 1000 items?

3. The cost function is $C(q) = 1000 + 20q$. Find the marginal cost to produce the 200th unit and the average cost of producing 200 units.

4. The graph of a cost function is given in Figure 4.64.
   (a) At $q = 25$, estimate the following quantities and represent your answers graphically.
   (i) Average cost  
   (ii) Marginal cost
   (b) At approximately what value of $q$ is average cost minimized?

5. Graph the average cost function corresponding to the total cost function shown in Figure 4.65.
6. For each cost function in Figure 4.66, is there a value of \( q \) at which average cost is minimized? If so, approximately where? Explain your answer.

![Figure 4.66](image)

7. You are the manager of a firm that produces slippers that sell for $20. You are producing 1200 slippers each month, at an average cost of $2 per slipper. The marginal cost at a production level of 1200 is $3 per slipper.

(a) Are you making or losing money?
(b) Will increasing production increase or decrease your average cost? Your profit?
(c) Would you recommend that production be increased or decreased?

8. The total cost of production, in thousands of dollars, is
   \[ C(q) = q^3 - 12q^2 + 60q, \]
   where \( q \) is in thousands and \( 0 \leq q \leq 8 \).

(a) Graph \( C(q) \). Estimate visually the quantity at which average cost is minimized.
(b) Determine analytically the exact value of \( q \) at which average cost is minimized.

9. The average cost per item to produce \( q \) items is given by
   \[ a(q) = 0.01q^2 - 0.6q + 13, \quad \text{for} \quad q > 0. \]

(a) What is the total cost, \( C(q) \), of producing \( q \) goods?
(b) What is the minimum marginal cost? What is the practical interpretation of this result?
(c) At what production level is the average cost a minimum? What is the lowest average cost?
(d) Compute the marginal cost at \( q = 30 \). How does this relate to your answer to part (c)? Explain this relationship both analytically and in words.

10. The marginal cost at a production level of 2000 units of an item is $10 per unit and the average cost of producing 2000 units is $15 per unit. If the production level were increased slightly above 2000, would the following quantities increase or decrease, or is it impossible to tell?

(a) Average cost \( b \)
(b) Profit \( mg \)

11. An agricultural worker in Uganda is planting clover to increase the number of bees making their home in the region. There are 100 bees in the region naturally, and for every acre put under clover, 20 more bees are found in the region.

(a) Draw a graph of the total number, \( N(x) \), of bees as a function of \( x \), the number of acres devoted to clover.

(b) Explain, both geometrically and algebraically, the shape of the graph of:
   (i) The marginal rate of increase of the number of bees with acres of clover, \( N'(x) \).
   (ii) The average number of bees per acre of clover, \( N(x)/x \).

12. A developer has recently purchased a laundromat and an adjacent factory. For years, the laundromat has taken pains to keep the smoke from the factory from soiling the air used by its clothes dryers. Now that the developer owns both the laundromat and the factory, she could install filters in the factory’s smokestacks to reduce the emission of smoke, instead of merely protecting the laundromat from it. The cost of filters for the factory and the cost of protecting the laundromat against smoke depend on the number of filters used, as shown in the table.

<table>
<thead>
<tr>
<th>Number of filters</th>
<th>Total cost of filters</th>
<th>Total cost of protecting laundromat from smoke</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0</td>
<td>$127</td>
</tr>
<tr>
<td>1</td>
<td>$5</td>
<td>$63</td>
</tr>
<tr>
<td>2</td>
<td>$11</td>
<td>$31</td>
</tr>
<tr>
<td>3</td>
<td>$18</td>
<td>$15</td>
</tr>
<tr>
<td>4</td>
<td>$26</td>
<td>$6</td>
</tr>
<tr>
<td>5</td>
<td>$35</td>
<td>$3</td>
</tr>
<tr>
<td>6</td>
<td>$45</td>
<td>$0</td>
</tr>
<tr>
<td>7</td>
<td>$56</td>
<td>$0</td>
</tr>
</tbody>
</table>

(a) Make a table which shows, for each possible number of filters (0 through 7), the marginal cost of the filter, the average cost of the filters, and the marginal savings in protecting the laundromat from smoke.
(b) Since the developer wishes to minimize the total costs to both her businesses, what should she do? Use the table from part (a) to explain your answer.
(c) What should the developer do if, in addition to the cost of the filters, the filters must be mounted on a rack which costs $100?
(d) What should the developer do if the rack costs $50?

13. Figure 4.67 shows the average cost, \( a(q) = b + mq \).

(a) Show that \( C'(q) = b + 2mq \).
(b) Graph the marginal cost \( C'(q) \).

![Figure 4.67](image)
14. Show analytically that if marginal cost is less than average cost, then the derivative of average cost with respect to quantity satisfies \( a'(q) < 0 \).

15. Show analytically that if marginal cost is greater than average cost, then the derivative of average cost with respect to quantity satisfies \( a'(q) > 0 \).

16. A reasonably realistic model of a firm’s costs is given by the short-run Cobb-Douglas cost curve

\[
C(q) = Kq^{1/a} + F,
\]

where \( a \) is a positive constant, \( F \) is the fixed cost, and \( K \) measures the technology available to the firm.

(a) Show that \( C \) is concave down if \( a > 1 \).

(b) Assuming that \( a < 1 \), find what value of \( q \) minimizes the average cost.

4.6 ELASTICITY OF DEMAND

The sensitivity of demand to changes in price varies with the product. For example, a change in the price of light bulbs may not affect the demand for light bulbs much, because people need light bulbs no matter what their price. However, a change in the price of a particular make of car may have a significant effect on the demand for that car, because people can switch to another make.

Elasticity of Demand

We want to find a way to measure this sensitivity of demand to price changes. Our measure should work for products as diverse as light bulbs and cars. The prices of these two items are so different that it makes little sense to talk about absolute changes in price: Changing the price of light bulbs by $1 is a substantial change, whereas changing the price of a car by $1 is not. Instead, we use the percent change in price. How, for example, does a 1% increase in price affect the demand for the product?

Let \( \Delta p \) denote the change in the price \( p \) of a product and \( \Delta q \) denote the corresponding change in quantity \( q \) demanded. The percent change in price is \( \Delta p/p \) and the percent change in demand is \( \Delta q/q \). Notice that \( \Delta p \) and \( \Delta q \) usually have opposite signs (because increasing the price decreases demand). Then the effect of a price change on demand is measured by the absolute value of the ratio

\[
\frac{\text{Percent change in demand}}{\text{Percent change in price}} = \left| \frac{\Delta q/q}{\Delta p/p} \right| = \left| \frac{\Delta q}{\Delta p} \cdot \frac{p}{q} \right| = \left| \frac{p}{q} \cdot \frac{\Delta q}{\Delta p} \right|
\]

For small changes in \( p \), we approximate \( \Delta q/\Delta p \) by the derivative \( dq/dp \). We define:

The elasticity of demand\(^8\) for a product, \( E \), is given approximately by

\[
E \approx \left| \frac{\Delta q/q}{\Delta p/p} \right|, \quad \text{or exactly by} \quad E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right|.
\]

Increasing the price of an item by 1% causes a drop of approximately \( E \)% in the quantity of goods demanded. For small changes, \( \Delta p \), in price,

\[
\frac{\Delta q}{q} \approx -E \frac{\Delta p}{p}.
\]

If \( E > 1 \), a 1% increase in price causes demand to drop by more than 1%, and we say that demand is elastic. If \( 0 \leq E < 1 \), a 1% increase in price causes demand to drop by less than 1%, and we say that demand is inelastic. In general, a larger elasticity causes a larger percent change in demand for a given percent change in price.

\(^8\)When it is necessary to distinguish it from other elasticities, this quantity is called the elasticity of demand with respect to price, or the price elasticity of demand.
Problems for Section 4.6

1. The elasticity of a good is $E = 0.5$. What is the effect on demand of:
   (a) A 3% price increase? (b) A 3% price decrease?

2. The elasticity of a good is $E = 2$. What is the effect on demand of:
   (a) A 3% price increase? (b) A 3% price decrease?

3. What is the elasticity for peaches in Table 4.5? Explain what this number tells you about the effect of price increases on the demand for peaches. Is the demand for peaches elastic or inelastic? Is this what you expect? Explain.

4. What is the elasticity for potatoes in Table 4.5? Explain what this number tells you about the effect of price increases on the demand for potatoes. Is the demand for potatoes elastic or inelastic? Is this what you expect? Explain.

5. Would you expect the demand for high definition television sets to be elastic or inelastic? Explain.

6. There are many brands of laundry detergent. Would you expect elasticity of demand for any particular brand to be high or low? Explain.

7. There is only one company offering local telephone service in a town. Would you expect elasticity of demand for telephone service to be high or low? Explain.

8. The demand curve for a product is given by $q = 200 - 2p^2$. Find the elasticity of demand when the price is $5$. Is the demand inelastic or elastic, or neither?

9. The demand curve for a product is given by $p = 90 - 10q$. Find the elasticity of demand when $p = 50$. If this price rises by 2%, calculate the corresponding percentage change in demand.

10. School organizations raise money by selling candy door to door. The table shows $p$, the price of the candy, and $q$, the quantity sold at that price.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$1.00$</th>
<th>$1.25$</th>
<th>$1.50$</th>
<th>$1.75$</th>
<th>$2.00$</th>
<th>$2.25$</th>
<th>$2.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>2765</td>
<td>2440</td>
<td>1980</td>
<td>1660</td>
<td>1175</td>
<td>800</td>
<td>430</td>
</tr>
</tbody>
</table>

   (a) Estimate the elasticity of demand at a price of $1.00$. At this price, is the demand elastic or inelastic?
   (b) Estimate the elasticity at each of the prices shown. What do you notice? Give an explanation for why this might be so.
   (c) At approximately what price is elasticity equal to 1?
   (d) Find the total revenue at each of the prices shown. Confirm that the total revenue appears to be maximized at approximately the price where $E = 1$.

11. What are the units of elasticity if:
   (a) Price $p$ is in dollars and quantity $q$ is in tons?
   (b) Price $p$ is in yen and quantity $q$ is in liters?
   (c) What can you conclude in general?

12. The demand for yams is given by $q = 5000 - 10p^2$, where $q$ is in pounds of yams and $p$ is the price of a pound of yams.

   (a) If the current price of yams is $2$ per pound, how many pounds will be sold?
   (b) Is the demand at $2$ elastic or inelastic? Is it more accurate to say “People want yams and will buy them no matter what the price” or “Yams are a luxury item and people will stop buying them if the price gets too high”?

13. The demand for yams is given in Problem 12.

   (a) At a price of $2$ per pound, what is the total revenue for the yam farmer?
   (b) Write revenue as a function of price, and then find the price that maximizes revenue.
   (c) What quantity is sold at the price you found in part (b), and what is the total revenue?
   (d) Show that $E = 1$ at the price you found in part (b).

14. It has been estimated that the elasticity of demand for slaves in the American South before the civil war was equal to 0.86 (fairly high) in the cities and equal to 0.05 (very low) in the countryside.\(^{10}\)

   (a) Why might this be?
   (b) Where do you think the staunchest defenders of slavery were from, the cities or the countryside?

15. Find the exact price that maximizes revenue for sales of the product in Example 2.

16. If $E = 2$ for all prices $p$, how can you maximize revenue?

17. If $E = 0.5$ for all prices $p$, how can you maximize revenue?

18. (a) If the demand equation is $pq = k$ for a positive constant $k$, compute the elasticity of demand.
   (b) Explain the answer to part (a) in terms of the revenue function.

19. Show that a demand equation $q = k/p^r$, where $r$ is a positive constant, gives constant elasticity $E = r$.

20. A linear demand function is given in Figure 4.68. Economists compute elasticity of demand $E$ for any quantity $q_0$ using the formula

   \[
   E = d_1/d_2, \]

   where $d_1$ and $d_2$ are the vertical distances shown in Figure 4.68.

   (a) Explain why this formula works.

---

(b) Determine the prices, \( p \), at which

(i) \( E > 1 \)
(ii) \( E < 1 \)
(iii) \( E = 1 \)

[Hint: Combine the result of Problem 21 with the fact that profit is maximized when \( MR = MC \).]

23. Elasticity of cost with respect to quantity is defined as \( E_{C,q} = q/C \cdot dC/dq \).

(a) What does this elasticity tell you about sensitivity of cost to quantity produced?
(b) Show that \( E_{C,q} = \) Marginal cost/Average cost.

24. If \( q \) is the quantity of chicken demanded as a function of the price \( p \) of beef, the cross-price elasticity of demand for chicken with respect to the price of beef is defined as \( E_{\text{cross}} = |p/q \cdot dq/dp| \). What does \( E_{\text{cross}} \) tell you about the sensitivity of the quantity of chicken bought to changes in the price of beef?

25. The income elasticity of demand for a product is defined as \( E_{\text{income}} = |l/q \cdot dq/dl| \) where \( q \) is the quantity demanded as a function of the income \( l \) of the consumer. What does \( E_{\text{income}} \) tell you about the sensitivity of the quantity of the product purchased to changes in the income of the consumer?

4.7 LOGISTIC GROWTH

In 1923, eighteen koalas were introduced to Kangaroo Island, off the coast of Australia.\(^{11}\) The koalas thrived on the island and their population grew to about 5000 in 1997. Is it reasonable to expect the population to continue growing exponentially? Since there is only a finite amount of space on the island, the population cannot grow without bound forever. Instead we expect that there is a maximum population that the island can sustain. Population growth with an upper bound can be modeled with a logistic or inhibited growth model.

Modeling the US Population

Population projections first became important to political philosophers in the late eighteenth century. As concern for scarce resources has grown, so has the interest in accurate population projections. In the US, the population is recorded every ten years by a census. The first such census was in 1790. Table 4.6 contains the census data from 1790 to 2000.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3.9</td>
<td>1850</td>
<td>23.1</td>
<td>1910</td>
<td>92.0</td>
<td>1960</td>
<td>179.3</td>
</tr>
<tr>
<td>1800</td>
<td>5.3</td>
<td>1860</td>
<td>31.4</td>
<td>1920</td>
<td>105.7</td>
<td>1970</td>
<td>203.3</td>
</tr>
<tr>
<td>1810</td>
<td>7.2</td>
<td>1870</td>
<td>38.6</td>
<td>1930</td>
<td>122.8</td>
<td>1980</td>
<td>226.5</td>
</tr>
<tr>
<td>1820</td>
<td>9.6</td>
<td>1880</td>
<td>50.2</td>
<td>1940</td>
<td>131.7</td>
<td>1990</td>
<td>248.7</td>
</tr>
<tr>
<td>1830</td>
<td>12.9</td>
<td>1890</td>
<td>62.9</td>
<td>1950</td>
<td>150.7</td>
<td>2000</td>
<td>281.4</td>
</tr>
<tr>
<td>1840</td>
<td>17.1</td>
<td>1900</td>
<td>76.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^ {11}\)Watertown Daily Times, April 18, 1997.
Example 4  Figure 4.80 shows dose-response curves for three different drugs used for the same purpose. Discuss the advantages and disadvantages of the three drugs.

![Dose-response curves for three drugs](image)

**Figure 4.80:** What are the advantages and disadvantages of each of these drugs?

Solution  Drugs A and B exhibit the same maximum response, while the maximum response of Drug C is significantly less; however all three drugs reach the minimum desired response. The potency of Drugs B and C (the dose required to reach desired effect) is significantly less than the potency of Drug A. (Potency, however, is a relatively unimportant characteristic of a drug, since a less potent drug can simply be given in larger doses.) Drug A has a steeper slope than either of the other two. Both Drugs A and B can exceed the maximum safe response. Thus, Drug C may be the preferred drug despite its lower maximum effect because it is the safest to administer.

Problems for Section 4.7

1. If \( t \) is in years since 1990, one model for the population of the world, \( P \), in billions, is

\[
P = \frac{40}{1 + 11e^{-0.08t}}.
\]

(a) What does this model predict for the maximum sustainable population of the world?

(b) Graph \( P \) against \( t \).

(c) According to this model, when will the earth’s population reach 20 billion? 39.9 billion?

2. A rumor spreads among a group of 400 people. The number of people, \( N(t) \), who have heard the rumor by time \( t \) in hours since the rumor started to spread can be approximated by a function of the form

\[
N(t) = \frac{400}{1 + 399e^{-0.4t}}.
\]

(a) Find \( N(0) \) and interpret it.

(b) How many people will have heard the rumor after 2 hours? After 10 hours?

(c) Graph \( N(t) \).

(d) Approximately how long will it take until half the people have heard the rumor? Virtually everyone?

(e) Approximately when is the rumor spreading fastest?

3. The rate of sales of an automobile anti-theft device are given in the following table.

(a) When is the point of diminishing returns reached?

(b) What are the total sales at this point?

<table>
<thead>
<tr>
<th>Months</th>
<th>Sales per month</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>140</td>
</tr>
<tr>
<td>2</td>
<td>520</td>
</tr>
<tr>
<td>3</td>
<td>680</td>
</tr>
<tr>
<td>4</td>
<td>750</td>
</tr>
<tr>
<td>5</td>
<td>700</td>
</tr>
<tr>
<td>6</td>
<td>550</td>
</tr>
</tbody>
</table>

4. The following table shows the total sales, in thousands, since a new game was brought to market.

(a) Plot this data and mark on your plot the point of diminishing returns.

(b) Predict total possible sales of this game, using the point of diminishing returns.

<table>
<thead>
<tr>
<th>Month</th>
<th>Sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>5.5</td>
</tr>
<tr>
<td>4</td>
<td>9.6</td>
</tr>
<tr>
<td>6</td>
<td>18.2</td>
</tr>
<tr>
<td>8</td>
<td>31.8</td>
</tr>
<tr>
<td>10</td>
<td>42.0</td>
</tr>
<tr>
<td>12</td>
<td>50.8</td>
</tr>
</tbody>
</table>

5. Write a paragraph explaining why sales of a new product often follow a logistic curve. Explain the benefit to the company of watching for the point of diminishing returns.

6. Investigate the effect of the parameter \( C \) on the logistic curve \( P = \frac{10}{1 + Ce^{-t}} \). Substitute several values for \( C \) and explain, with a graph and with words, the effect of \( C \) on the graph.
7. Figure 4.81 shows the spread of the Code-red computer virus during July 2001. Most of the growth took place starting at midnight on July 19; on July 20, the virus attacked the White House, trying (unsuccessfully) to knock its site off-line. The number of computers infected by the virus is a logistic function of time.

(a) Estimate \( \lim_{t \to \infty} f(t) \). What does this limit represent in terms of Code-red?

(b) Estimate the value of \( t \) at which \( f''(t) = 0 \). Estimate the value of \( n \) at this time.

(c) What does the answer to part (b) tell us about Code-red?

(d) How are the answers to parts (a) and (b) related?

8. (a) Draw a logistic curve. Label the carrying capacity \( L \) and the point of diminishing returns \( t_0 \).

(b) Draw the derivative of the logistic curve. Mark the point \( t_0 \) on the horizontal axis.

(c) A company keeps track of the rate of sales (for example, sales per week) rather than total sales. Explain how the company can tell on a graph of rate of sales when the point of diminishing returns is reached.

9. The Tojolobal Mayan Indian community in Southern Mexico has available a fixed amount of land. The proportion, \( P \), of land in use for farming \( t \) years after 1935 is modeled with the logistic function

\[
P = \frac{1}{1 + 3e^{-0.0275t}}.
\]

(a) What proportion of the land was in use for farming in 1935?

(b) What is the long run prediction of this model?

(c) When was half the land in use for farming?

(d) When is the proportion of land used for farming increasing most rapidly?

10. In the spring of 2003, SARS (Severe Acute Respiratory Syndrome) spread rapidly in several Asian countries and Canada. Table 4.9 gives the total number, \( P \), of SARS cases reported in Hong Kong by day \( t \), where \( t = 0 \) is March 17, 2003.

(a) Find the average rate of change of \( P \) for each interval in Table 4.9.

(b) In early April 2003, there was fear that the disease would spread at an ever-increasing rate for a long time. What is the earliest date by which epidemiologists had evidence to indicate that the rate of new cases had begun to slow?

(c) Explain why an exponential model for \( P \) is not appropriate.

(d) It turns out that a logistic model fits the data well. Estimate the value of \( t \) at the inflection point. What limiting value of \( P \) does this point predict?

(e) The best-fitting logistic function for this data turns out to be

\[
P = \frac{1760}{1 + 17.53e^{-0.1408t}}.
\]

What limiting value of \( P \) does this function predict?

| Table 4.9 Total number of SARS cases in Hong Kong by day \( t \) (where \( t = 0 \) is March 17, 2003) |
|---|---|---|---|---|---|---|
| \( t \) (hours since midnight) | \( P \) | \( t \) (hours since midnight) | \( P \) | \( t \) (hours since midnight) | \( P \) | \( t \) (hours since midnight) | \( P \) |
| 8 | 95 | 16 | 26 | 1108 | 24 | 54 | 1674 | 32 | 75 | 1739 |
| 5 | 222 | 33 | 33 | 1358 | 61 | 1710 | 81 | 1750 |
| 12 | 470 | 40 | 40 | 1527 | 68 | 1724 | 87 | 1755 |
| 19 | 800 | 47 | 1621 | 19 | 800 |

11. Substitute \( t = 0, 10, 20, \ldots, 70 \) into the exponential function used in this section to model the US population 1790–1860. Compare the predicted values of the population with the actual values.

12. In this section, a logistic function was used to model the US population. Use this function to predict the US population in each of the census years from 1790–1940. Compare the predicted and actual values.

13. A curve representing the total number of people, \( P \), infected with a virus often has the shape of a logistic curve of the form \( P = \frac{L}{1 + Ce^{-kt}} \), with time \( t \) in weeks. Suppose that 10 people originally have the virus and that in the early stages the number of people infected is increasing approximately exponentially, with a continuous growth rate of 1.78. It is estimated that, in the long run, approximately 5000 people will become infected.

(a) What proportion of the land was in use for farming in 1935?

(b) What is the long run prediction of this model?

(c) When was half the land in use for farming?

(d) When is the proportion of land used for farming increasing most rapidly?

10. In the spring of 2003, SARS (Severe Acute Respiratory Syndrome) spread rapidly in several Asian countries and Canada. Table 4.9 gives the total number, \( P \), of SARS cases reported in Hong Kong by day \( t \), where \( t = 0 \) is March 17, 2003.

(a) Find the average rate of change of \( P \) for each interval in Table 4.9.

11. Substitute \( t = 0, 10, 20, \ldots, 70 \) into the exponential function used in this section to model the US population 1790–1860. Compare the predicted values of the population with the actual values.

12. In this section, a logistic function was used to model the US population. Use this function to predict the US population in each of the census years from 1790–1940. Compare the predicted and actual values.

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(a) What should we use for the parameters \( k \) and \( L \)?

(b) Use the fact that when \( t = 0 \), we have \( P = 10 \), to find \( C \).

(c) Now that you have estimated \( L \), \( k \), and \( C \), what is the logistic function you are using to model the data? Graph this function.

(d) Estimate the length of time until the rate at which people are becoming infected starts to decrease. What is the value of \( P \) at this point?

\[ \text{Adapted from J. S. Thomas and M. C. Robbins, "The Limits to Growth in a Tojolobal Maya Ejido," Geoscience and Man 26, pp. 9–16, (Baton Rouge: Geoscience Publications, 1988).} \]

\[ \text{Website: www.who.int/csr/country/en, accessed July 13, 2003.} \]
14. If $R$ is percent of maximum response and $x$ is dose in mg, the dose-response curve for a drug is given by

$$R = \frac{100}{1 + 100e^{-0.1x}}.$$  

(a) Graph this function.
(b) What dose corresponds to a response of 50% of the maximum? This is the inflection point, at which the response is increasing the fastest.
(c) For this drug, the minimum desired response is 20% and the maximum safe response is 70%. What range of doses is both safe and effective for this drug?

15. Dose-response curves for three different products are given in Figure 4.82.

(a) For the desired response, which drug requires the largest dose? The smallest dose?
(b) Which drug has the largest maximum response? The smallest?
(c) Which drug is the safest to administer? Explain.

![Figure 4.82](image)

16. A dose-response curve is given by $R = f(x)$, where $R$ is percent of maximum response and $x$ is the dose of the drug in mg. The curve has the shape shown in Figure 4.79 on page 218. The inflection point is at $(15, 50)$ and $f'(15) = 11$.

(a) Explain what $f'(15)$ tells you in terms of dose and response for this drug.
(b) Is $f'(10)$ greater than or less than 11? Is $f'(20)$ greater than or less than 11? Explain.

17. Explain why it is safer to use a drug for which the derivative of the dose-response curve is smaller.

18. In Figure 4.83, what range of doses appears to be both safe and effective for 99% of all patients?

![Figure 4.83](image)

19. In Figure 4.84, discuss the possible outcomes and what percent of patients fall in each outcome when 50 mg of the drug is administered.

![Figure 4.84](image)

### 4.8 THE SURGE FUNCTION AND DRUG CONCENTRATION

#### Nicotine in the Blood

When a person smokes a cigarette, the nicotine from the cigarette enters the person’s body through the lungs, is absorbed into the blood, and spreads throughout the body. Most cigarettes contain between 0.5 and 2.0 mg of nicotine; approximately 20% (between 0.1 and 0.4 mg) is actually inhaled and absorbed into the person’s bloodstream. As the nicotine leaves the blood, the smoker feels the need for another cigarette. The half-life of nicotine in the bloodstream is about two hours. The lethal dose is considered to be about 60 mg.
Solution
The minimum effective concentration on the drug concentration curve is plotted as a dotted horizontal line at 4 ng/ml. See Figure 4.94. We see that the drug becomes effective almost immediately and ceases to be effective after about four months. Doses should be given about every four months.

Although the dosage interval is four months, notice that it takes ten months after injections are discontinued for Depo-Provera to be entirely eliminated from the body. Fertility during that period is unpredictable.

Problems for Section 4.8

1. If time, $t$, is in hours and concentration, $C$, is in ng/ml, the drug concentration curve for a drug is given by

   $C = 12.4te^{-0.2t}$.

   (a) Graph this curve.
   (b) How many hours does it take for the drug to reach its peak concentration? What is the concentration at that time?
   (c) If the minimum effective concentration is 10 ng/ml, during what time period is the drug effective?
   (d) Complications can arise whenever the level of the drug is above 4 ng/ml. How long must a patient wait before being safe from complications?

2. Figure 4.95 shows drug concentration curves for anhydrous ampicillin for newborn babies and adults.\(^{21}\) Discuss the differences between newborns and adults in the absorption of this drug.

3. If $t$ is in hours, the drug concentration curve for a drug is given by $C = 17.2te^{-0.4t}$ ng/ml. The minimum effective concentration is 10 ng/ml.
   (a) If the second dose of the drug is to be administered when the first dose becomes ineffective, when should the second dose be given?
   (b) If you want the onset of effectiveness of the second dose to coincide with termination of effectiveness of the first dose, when should the second dose be given?

4. Absorption of different forms of the antibiotic erythromycin may be increased, decreased, delayed or not affected by food. Figure 4.96 shows the drug concentration levels of erythromycin in healthy, fasting human volunteers who received single oral doses of 500 mg erythromycin tablets, together with either large (250 ml) or small (20 ml) accompanying volumes of water.\(^{22}\) Discuss the effect of the water on the concentration of erythromycin in the blood. How are the peak concentration and the time to reach peak concentration affected? When does the effect of the volume of water wear off?

5. Hydrocodone bitartrate is a cough suppressant usually administered in a 10 mg oral dose. The peak concentration of the drug in the blood occurs 1.3 hours after consumption and the peak concentration is 23.6 ng/ml. Draw the drug concentration curve for hydrocodone bitartrate.

6. If $t$ is in minutes since the drug was administered, the concentration, $C(t)$ in ng/ml, of a drug in a patient’s bloodstream is given by

   $C(t) = 20te^{-0.031}$.

   (a) How long does it take for the drug to reach peak concentration? What is the peak concentration?
   (b) What is the concentration of the drug in the body after 15 minutes? After an hour?
   (c) If the minimum effective concentration is 10 ng/ml, when should the next dose be administered?
7. Figure 4.85 on page 222 shows the concentration of nicotine in the blood during and after smoking a cigarette. Figure 4.97 shows the concentration of nicotine in the blood during and after using chewing tobacco or nicotine gum. (The chewing occurred during the first 30 minutes and the experimental data shown represent the average values for ten patients.) Compare the three nicotine concentration curves (for cigarettes, chewing tobacco and nicotine gum) in terms of peak concentration, the time until peak concentration, and the rate at which the nicotine is eliminated from the bloodstream.

8. The method of administering a drug can have a strong influence on the drug concentration curve. Figure 4.98 shows drug concentration curves for penicillin following various routes of administration. Three milligrams per kilogram of body weight were dissolved in water and administered intravenously (IV), intramuscularly (IM), subcutaneously (SC), and orally (PO). The same quantity of penicillin dissolved in oil was administered intramuscularly (P-IM). The minimum effective concentration (MEC) is labeled on the graph.

(a) Which method reaches peak concentration the fastest? The slowest?

(b) Which method has the largest peak concentration? The smallest?
(c) Which method wears off the fastest? The slowest?
(d) Which method has the longest effective duration? The shortest?
(e) When penicillin is administered orally, for approximately what time interval is it effective?

9. Figure 4.99 shows the plasma levels of canrenone in a healthy volunteer after a single oral dose of spironolactone given on a fasting stomach and together with a standardized breakfast. (Spironolactone is a diuretic agent that is partially converted into canrenone in the body.) Discuss the effect of food on peak concentration and time to reach peak concentration. Is the effect of the food strongest during the first 8 hours, or after 8 hours?

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25 Welling & Tse, Pharmacokinetics of Cardiovascular, Central Nervous System, and Antimicrobial Drugs, (The Royal Society of Chemistry, 1985).
10. Figure 4.100 shows drug concentration curves after oral administration of 0.5 mg of four digoxin products. All the tablets met current USP standards of potency, disintegration time, and dissolution rate.26

(a) Discuss differences and similarities in the peak concentration and the time to reach peak concentration.
(b) Give possible values for minimum effective concentration and maximum safe concentration that would make Product C or Product D the preferred drug.
(c) Give possible values for minimum effective concentration and maximum safe concentration that would make Product A the preferred drug.

11. Figure 4.101 shows a graph of the percentage of drug dissolved against time for four tetracycline products A, B, C, and D. Figure 4.102 shows the drug concentration curves for the same four tetracycline products.27 Discuss the effect of dissolution rate on peak concentration and time to reach peak concentration.

12. Let $b = 1$, and graph $C = ate^{-bt}$ using different values for $a$. Explain the effect of the parameter $a$.

CHAPTER SUMMARY

- Using the first derivative
  Critical points, local maxima and minima
- Using the second derivative
  Inflection points, concavity
- Optimization
  Global maxima and minima
- Maximizing profit and revenue
- Average cost
  Minimizing average cost
- Elasticity
- Families of functions
  Parameters. The surge function, drug concentration curves. The logistic function, carrying capacity, point of diminishing returns.

REVIEW PROBLEMS FOR CHAPTER FOUR

For Problems 1–2, indicate all critical points on the given graphs. Determine which correspond to local minima, local maxima, global minima, global maxima, or none of these. (Note that the graphs are on closed intervals.)