Problems 14–15 show the graph of a derivative function \( f' \). Indicate on a sketch the \( x \)-values that are critical points of the function \( f \) itself. Identify each critical point as a local maximum, a local minimum, or neither.

14. \( f'(x) \) 
15. \( f'(x) \)

16. Figure 4.13 is a graph of \( f' \). For what values of \( x \) does \( f \) have a local maximum? A local minimum?

17. On the graph of \( f' \) in Figure 4.14, indicate the \( x \)-values that are critical points of the function \( f \) itself. Are they local maxima, local minima, or neither?

18. The derivative of \( f(t) \) is given by \( f'(t) = t^3 - 6t^2 + 8t \) for \( 0 \leq t \leq 5 \). Graph \( f'(t) \), and describe how the function \( f(t) \) changes over the interval \( t = 0 \) to \( t = 5 \). When is \( f(t) \) increasing and when is it decreasing? Where does \( f(t) \) have a local maximum and where does it have a local minimum?

19. Consumer demand for a certain product is changing over time, and the rate of change of this demand, \( f'(t) \), in units/week, is given, in week \( t \), in the following table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(t) )</td>
<td>12</td>
<td>10</td>
<td>4</td>
<td>-2</td>
<td>-3</td>
<td>-1</td>
<td>3</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) When is the demand for this product increasing? When is it decreasing?
(b) Approximately when is demand at a local maximum? A local minimum?

20. Suppose \( f \) has a continuous derivative whose values are given in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>-2</td>
<td>-5</td>
<td>-3</td>
<td>-1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

(a) Estimate the \( x \)-coordinates of critical points of \( f \) for \( 0 \leq x \leq 10 \).
(b) For each critical point, indicate if it is a local maximum of \( f \), local minimum, or neither.

21. The function \( f(x) = x^4 - 4x^3 + 8x \) has a critical point at \( x = 1 \). Use the second derivative test to identify it as a local maximum or local minimum.

22. Find and classify the critical points of \( f(x) = x^3(1-x)^4 \) as local maxima and minima.

In Problems 23–24, find constants \( a \) and \( b \) so that the minimum for the parabola \( f(x) = x^2 + ax + b \) is at the given point. [Hint: Begin by finding the critical point in terms of \( a \).]

23. \((3, 5)\)
24. \((-2, -3)\)

25. Sketch several members of the family \( y = x^3 - ax^2 \) on the same axes. Discuss the effect of the parameter \( a \) on the graph. Find all critical points for this function.

26. For what values of \( a \) and \( b \) does \( f(x) = a(x - b \ln x) \) have a local minimum at the point \((2, 5)\)? Figure 4.15 shows a graph of \( f(x) \) with \( a = 1 \) and \( b = 1 \).

27. Find the value of \( a \) so that the function \( f(x) = xe^{ax} \) has a critical point at \( x = 3 \).

28. Assume \( f \) has a derivative everywhere and has just one critical point, at \( x = 3 \). In parts (a)–(d), you are given additional conditions. In each case decide whether \( x = 3 \) is a local maximum, a local minimum, or neither. Explain your reasoning. Sketch possible graphs for all four cases.

(a) \( f'(1) = 3 \) and \( f'(5) = -1 \)
(b) \( f(x) \to \infty \) as \( x \to \infty \) and as \( x \to -\infty \)
(c) \( f(1) = 1, f(2) = 2, f(4) = 4, f(5) = 5 \)
(d) \( f'(2) = -1, f'(3) = 1, f(x) \to 3 \) as \( x \to \infty \)

29. (a) On a computer or calculator, graph \( f(\theta) = \theta - \sin \theta \).
Can you tell whether the function has any zeros in the interval \( 0 \leq \theta \leq 1? \)
(b) Find \( f' \). What does the sign of \( f' \) tell you about the zeros of \( f \) in the interval \( 0 \leq \theta \leq 1 \)?
Problems for Section 4.2

In Problems 1–4, indicate the approximate locations of all inflection points. How many inflection points are there?

1. \[ f(t) \]
2. \[ f(x) \]
3. \[ f(x) \]
4. \[ f(x) \]

5. Graph a function with only one critical point (at \( x = 5 \)) and one inflection point (at \( x = 10 \)). Label the critical point and the inflection point on your graph.

6. (a) Graph a polynomial with two local maxima and two local minima.
   (b) What is the least number of inflection points this function must have? Label the inflection points.

7. Graph a function which has a critical point and an inflection point at the same place.

8. When I got up in the morning I put on only a light jacket because, although the temperature was dropping, it seemed that the temperature would not go much lower. But I was wrong. Around noon a northerly wind blew up and the temperature began to drop faster and faster. The worst was around 6 pm when, fortunately, the temperature started going back up.

   (a) When was there a critical point in the graph of temperature as a function of time?
   (b) When was there an inflection point in the graph of temperature as a function of time?

9. During a flood, the water level in a river first rose faster and faster, then rose more and more slowly until it reached its highest point, then went back down to its pre-flood level. Consider water depth as a function of time.

   (a) Is the time of highest water level a critical point or an inflection point of this function?
   (b) Is the time when the water first began to rise more slowly a critical point or an inflection point?

10. For \( f(x) = x^3 - 18x^2 - 10x + 6 \), find the inflection point algebraically. Graph the function with a calculator or computer and confirm your answer.

In each of Problems 11–20, use the first derivative to find all critical points and use the second derivative to find all inflection points. Use a graph to identify each critical point as a local maximum, a local minimum, or neither.

11. \( f(x) = x^2 - 5x + 3 \)
12. \( f(x) = x^3 - 3x + 10 \)
13. \( f(x) = 2x^3 + 3x^2 - 36x + 5 \)
14. \( f(x) = \frac{x^3}{6} + \frac{x^2}{4} - x + 2 \)
15. \( f(x) = x^4 - 2x^2 \)
16. \( f(x) = 3x^4 - 4x^3 + 6 \)
17. \( f(x) = x^4 - 8x^2 + 5 \)
18. \( f(x) = x^4 - 4x^3 + 10 \)
19. \( f(x) = x^5 - 5x^4 + 35 \)
20. \( f(x) = 3x^5 - 5x^3 \)

21. Find the inflection points of \( f(x) = x^4 + x^3 - 3x^2 + 2 \).

For Problems 22–25, sketch a possible graph of \( y = f(x) \), using the given information about the derivatives \( y' = f'(x) \) and \( y'' = f''(x) \). Assume that the function is defined and continuous for all real \( x \).

22. \( y' = 0 \)
    \( y' > 0 \)
    \( y' < 0 \)
    \( y'' = 0 \)
    \( y'' > 0 \)
    \( y'' < 0 \)

23. \( y' < 0 \)
    \( y' > 0 \)
    \( y'' = 0 \)
    \( y'' > 0 \)
    \( y'' < 0 \)

24. \( y' = 0 \)
    \( y' > 0 \)
    \( y' < 0 \)
    \( y'' = 0 \)
    \( y'' > 0 \)
    \( y'' < 0 \)

25. \( y' = 2 \)
    \( y' > 0 \)
    \( y'' = 0 \)
    \( y'' > 0 \)
    \( y'' < 0 \)
26. In 1774, Captain James Cook left 10 rabbits on a small Pacific island. The rabbit population is approximated by

\[ P(t) = \frac{2000}{1 + e^{5.3 - 0.4t}} \]

with \( t \) measured in years since 1774. Using a calculator or computer:

(a) Graph \( P \). Does the population level off?

(b) Estimate when the rabbit population grew most rapidly. How large was the population at that time?

(c) Find the inflection point on the graph and explain its significance for the rabbit population.

(d) What natural causes could lead to the shape of the graph of \( P \)?

27. (a) Water is flowing at a constant rate (i.e., constant volume per unit time) into a cylindrical container standing vertically. Sketch a graph showing the depth of water against time.

(b) Water is flowing at a constant rate into a cone-shaped container standing on its point. Sketch a graph showing the depth of the water against time.

28. If water is flowing at a constant rate (i.e., constant volume per unit time) into the Grecian urn in Figure 4.25, sketch a graph of the depth of the water against time. Mark on the graph the time at which the water reaches the widest point of the urn.

31. The vase in Figure 4.28 is filled with water at a constant rate (i.e., constant volume per unit time).

(a) Graph \( y = f(t) \), the depth of the water, against time, \( t \). Show on your graph the points at which the concavity changes.

(b) At what depth is \( y = f(t) \) growing most quickly? Most slowly? Estimate the ratio between the growth rates at these two depths.

32. Assume that the polynomial \( f \) has exactly two local maxima and one local minimum, and that these are the only critical points of \( f \).

(a) Sketch a possible graph of \( f \).

(b) What is the largest number of zeros \( f \) could have?

(c) What is the least number of zeros \( f \) could have?

(d) What is the least number of inflection points \( f \) could have?

(e) What is the smallest degree \( f \) could have?

(f) Find a possible formula for \( f(x) \).

33. Indicate on Figure 4.29 approximately where the inflection points of \( f(x) \) are if the graph shows

(a) The function \( f(x) \)

(b) The derivative \( f'(x) \)

(c) The second derivative \( f''(x) \)
Problems for Section 4.3

For Problems 1–2, indicate all critical points on the given graphs. Which correspond to local minima, local maxima, global maxima, global minima, or none of these? (Note that the graphs are on closed intervals.)

1. [Graph]

2. [Graph]

3. A grapefruit is tossed straight up with an initial velocity of 50 ft/sec. The grapefruit is 5 feet above the ground when it is released. Its height at time \( t \) is given by

\[
y = -16t^2 + 50t + 5.
\]

How high does it go before returning to the ground?

4. For each interval, use Figure 4.37 to choose the statement that gives the location of the global maximum and global minimum of \( f \) on the interval.

   (a) \( 4 \leq x \leq 12 \)  
   (b) \( 11 \leq x \leq 16 \)  
   (c) \( 4 \leq x \leq 9 \)  
   (d) \( 8 \leq x \leq 18 \)

   (I) Maximum at right endpoint, minimum at left endpoint.
   (II) Maximum at right endpoint, minimum at critical point.
   (III) Maximum at left endpoint, minimum at right endpoint.
   (IV) Maximum at left endpoint, minimum at critical point.

In Problems 5–8, graph a function with the given properties.

5. Has local minimum and global minimum at \( x = 3 \) but no local or global maximum.

6. Has local minimum at \( x = 3 \), local maximum at \( x = 8 \), but no global maximum or minimum.

7. Has local and global minimum at \( x = 3 \), local and global maximum at \( x = 8 \).

8. Has no local or global maxima or minima.

9. True or false? Give an explanation for your answer. The global maximum of \( f(x) = x^2 \) on every closed interval is at one of the endpoints of the interval.

In Problems 10–13, sketch the graph of a function on the interval \( 0 \leq x \leq 10 \) with the given properties.

10. Has local minimum at \( x = 3 \), local maximum at \( x = 8 \), but global maximum and global minimum at the endpoints of the interval.

11. Has local and global maximum at \( x = 3 \), local and global minimum at \( x = 10 \).

12. Has local and global minimum at \( x = 3 \), local and global maximum at \( x = 8 \).

13. Has global maximum at \( x = 0 \), global minimum at \( x = 10 \), and no other local maxima or minima.

14. Plot the graph of \( f(x) = x^3 - e^x \) using a graphing calculator or computer to find all local and global maxima and minima for:

   (a) \( -1 \leq x \leq 4 \)  
   (b) \( -3 \leq x \leq 2 \)

15. For some positive constant \( C \), a patient’s temperature change, \( T \), due to a dose, \( D \), of a drug is given by

\[
T = \left( \frac{C}{2} - \frac{D}{3} \right) D^2.
\]

   (a) What dosage maximizes the temperature change?
   (b) The sensitivity of the body to the drug is defined as \( dT/dD \). What dosage maximizes sensitivity?
16. Figure 4.38 shows the rate at which photosynthesis is taking place in a leaf.

   (a) At what time, approximately, is photosynthesis proceeding fastest for $t \geq 0$?
   (b) If the leaf grows at a rate proportional to the rate of photosynthesis, for what part of the interval $0 \leq t \leq 200$ is the leaf growing? When is it growing fastest?

![Figure 4.38](image)

For each of the functions in Problems 17–21, do the following:

   (a) Find $f'$ and $f''$.
   (b) Find the critical points of $f$.
   (c) Find any inflection points of $f$.
   (d) Evaluate $f$ at its critical points and at the endpoints of the given interval. Identify local and global maxima and minima of $f$ in the interval.
   (e) Graph $f$.

17. $f(x) = x^3 - 3x^2 \quad (-1 \leq x \leq 3)$
18. $f(x) = 2x^3 - 9x^2 + 12x + 1 \quad (-0.5 \leq x \leq 3)$
19. $f(x) = x^3 - 3x^2 - 9x + 15 \quad (-5 \leq x \leq 4)$
20. $f(x) = x + \sin x \quad (0 \leq x \leq 2\pi)$
21. $f(x) = e^{-x} \sin x \quad (0 \leq x \leq 2\pi)$

In Problems 22–27, find the exact global maximum and minimum values of the function. The domain is all real numbers unless otherwise specified.

22. $g(x) = 4x - x^2 - 5$
23. $f(x) = x + 1/x$ for $x > 0$
24. $g(t) = t e^{-t}$ for $t > 0$
25. $f(x) = x - \ln x$ for $x > 0$
26. $f(t) = \frac{t}{1 + t^2}$
27. $f(t) = (\sin^2 t + 2) \cos t$
28. Figure 4.39 gives the derivative of $g(x)$ on $-2 \leq x \leq 2$.
   (a) Write a few sentences describing the behavior of $g(x)$ on this interval.
   (b) Does the graph of $g(x)$ have any inflection points? If so, give the approximate $x$-coordinates of their locations. Explain your reasoning.
   (c) What are the global maxima and minima of $g$ on $[-2, 2]$?
   (d) If $g(-2) = 5$, what do you know about $g(0)$ and $g(2)$? Explain.

![Figure 4.39](image)

29. When you cough, your windpipe contracts. The speed, $v$, with which air comes out depends on the radius, $r$, of your windpipe. If $R$ is the normal (rest) radius of your windpipe, then for $r \leq R$, the speed is given by:

$$v = a(R - r)^2$$

where $a$ is a positive constant.

What value of $r$ maximizes the speed?

30. The energy expended by a bird per day, $E$, depends on the time spent foraging for food per day, $F$ hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense. Find the foraging time that minimizes energy expenditure if

$$E = 0.25F + \frac{1.7}{F^2}.$$

31. Find the dimensions of the rectangle with perimeter 200 meters that has the largest area.

32. If you have 100 feet of fencing and want to enclose a rectangular area up against a long, straight wall, what is the largest area you can enclose?

33. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing $25$ per foot along three sides and fencing costing $10$ per foot along the fourth side. Find the minimum total cost.

34. A closed box has a fixed surface area $A$ and a square base with side $x$.
   (a) Find a formula for its volume, $V$, as a function of $x$.
   (b) Sketch a graph of $V$ against $x$.
   (c) Find the maximum value of $V$.

35. A square-bottomed box with a top has a fixed volume, $V$. What dimensions minimize the surface area?

---

36. On the west coast of Canada, crows eat whelks (a shellfish). To open the whelks, the crows drop them from the air onto a rock. If the shell does not smash the first time, the whelk is dropped again. The average number of drops, \( n \), needed when the whelk is dropped from a height of \( x \) meters is approximated by

\[
n(x) = 1 + \frac{27}{x^2}.
\]

(a) Give the total vertical distance the crow travels upward to open a whelk as a function of drop height, \( x \).

(b) Crows are observed to drop whelks from the height that minimizes the total vertical upward distance traveled per whelk. What is this height?

37. During a flu outbreak in a school of 763 children, the number of infected children, \( I \), was expressed in terms of the number of susceptible (but still healthy) children, \( S \), by the expression\(^6\)

\[
I = 192 \ln \left( \frac{S}{702} \right) - S + 763.
\]

What is the maximum possible number of infected children?

38. (a) Find the critical points of \( p(1 - p)^4 \).

(b) Classify the critical points as local maxima, local minima, or neither.

(c) What are the maximum and minimum values of \( p(1 - p)^4 \) on \( 0 \leq x \leq 1 \)?

39. An apple tree produces, on average, 400 kg of fruit each season. However, if more than 200 trees are planted per km\(^2\), crowding reduces the yield by 1 kg for each tree over 200.

(a) Express the total yield from one square kilometer as a function of the number of trees on it. Graph this function.

(b) How many trees should a farmer plant on each square kilometer to maximize yield?

40. The number of offspring in a population may not be a linear function of the number of adults. The Ricker curve, used to model fish populations, claims that \( y = axe^{-bx} \), where \( x \) is the number of adults, \( y \) is the number of offspring, and \( a \) and \( b \) are positive constants.

(a) Find and classify all critical points of the Ricker curve.

(b) Is there a global maximum? What does this imply about populations?

41. A chemical reaction converts substance \( A \) to substance \( Y \); the presence of \( Y \) catalyzes the reaction. At the start of the reaction, the quantity of \( A \) present is \( a \) grams. At time \( t \) seconds later, the quantity of \( Y \) present is \( y \) grams. The rate of the reaction, in grams/sec, is given by

\[
\text{Rate} = k(y(a - y)), \quad k \text{ is a positive constant.}
\]

(a) For what values of \( y \) is the rate nonnegative? Graph the rate against \( y \).

(b) For what values of \( y \) is the rate a maximum?

42. In a chemical reaction, substance \( A \) combines with substance \( B \) to form substance \( Y \). At the start of the reaction, the quantity of \( A \) present is \( a \) grams, and the quantity of \( B \) present is \( b \) grams. Assume \( a < b \). At time \( t \) seconds after the start of the reaction, the quantity of \( Y \) present is \( y \) grams. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

\[
\text{Rate} = k(a - y)(b - y), \quad k \text{ is a positive constant.}
\]

(a) For what values of \( y \) is the rate nonnegative? Graph the rate against \( y \).

(b) Use your graph to find the value of \( y \) at which the rate of the reaction is fastest.

43. The oxygen supply, \( S \), in the blood depends on the hematocrit, \( H \), the percentage of red blood cells in the blood:

\[
S = aHe^{-bH} \quad \text{for positive constants } a, b.
\]

(a) What value of \( H \) maximizes the oxygen supply? What is the maximum oxygen supply?

(b) How does increasing the value of the constants \( a \) and \( b \) change the maximum value of \( S \)?

44. The quantity of a drug in the bloodstream \( t \) hours after a tablet is swallowed is given, in mg, by

\[
q(t) = 20(e^{-t} - e^{-2t}).
\]

(a) How much of the drug is in the bloodstream at time \( t = 0 \)?

(b) When is the maximum quantity of drug in the bloodstream? What is that maximum?

(c) In the long run, what happens to the quantity?

45. When birds lay eggs, they do so in clutches of several at a time. When the eggs hatch, each clutch gives rise to a brood of baby birds. We want to determine the clutch size which maximizes the number of birds surviving to adulthood per brood. If the clutch is small, there are few baby birds in the brood; if the clutch is large, there are so many baby birds to feed that most die of starvation.


\(^6\) Data from Communicable Disease Surveillance Centre (UK), reported in “Influenza in a Boarding School”, British Medical Journal; March 4, 1978.

\(^7\) Data from C. M. Perreia and D. Lack reported by J. R. Krebs and N. B. Davies in An Introduction to Behavioural Ecology (Oxford: Blackwell, 1987).
The number of surviving birds per brood as a function of clutch size is shown by the benefit curve in Figure 4.40.7

(a) Estimate the clutch size which maximizes the number of survivors per brood.

(b) Suppose also that there is a biological cost to having a larger clutch: the female survival rate is reduced by large clutches. This cost is represented by the dotted line in Figure 4.40. If we take cost into account by assuming that the optimal clutch size in fact maximizes the vertical distance between the curves, what is the new optimal clutch size?

![Figure 4.40](image)

46. Let \( f(v) \) be the amount of energy consumed by a flying bird, measured in joules per second (a joule is a unit of energy), as a function of its speed \( v \) (in meters/sec). Let \( a(v) \) be the amount of energy consumed by the same bird, measured in joules per meter.

(a) Suggest a reason (in terms of the way birds fly) for the shape of the graph of \( f(v) \) in Figure 4.41.

(b) What is the relationship between \( f(v) \) and \( a(v) \)?

(c) Where is \( a(v) \) a minimum?

(d) Should the bird try to minimize \( f(v) \) or \( a(v) \) when it is flying? Why?

![Figure 4.41](image)

47. As an epidemic spreads through a population, the number of infected people, \( I \), is expressed as a function of the number of susceptible people, \( S \), by

\[
I = k \ln \left( \frac{S}{S_0} \right) - S + S_0 + I_0, \quad \text{for } k, S_0, I_0 > 0.
\]

(a) Find the maximum number of infected people.

(b) The constant \( k \) is a characteristic of the particular disease; the constants \( S_0 \) and \( I_0 \) are the values of \( S \) and \( I \) when the disease starts. Which of the following affects the maximum possible value of \( I \)? Explain.
   - The particular disease, but not how it starts.
   - How the disease starts, but not the particular disease.
   - Both the particular disease and how it starts.

48. The hypotenuse of a right triangle has one end at the origin and one end on the curve \( y = x^2/e^{-3x} \), with \( x \geq 0 \). One of the other two sides is on the \( x \)-axis, the other side is parallel to the \( y \)-axis. Find the maximum area of such a triangle. At what \( x \)-value does it occur?

49. A right triangle has one vertex at the origin and one vertex on the curve \( y = e^{-x/3} \) for \( 1 \leq x \leq 5 \). One of the two perpendicular sides is along the \( x \)-axis; the other is parallel to the \( y \)-axis. Find the maximum and minimum areas for such a triangle.

50. A person's blood pressure, \( p \), in millimeters of mercury (mm Hg) is given, for \( t \) in seconds, by

\[
p = 100 + 20 \sin(2.5 \pi t).
\]

(a) What are the maximum and minimum values of blood pressure?

(b) What is the interval between successive maxima?

(c) Show your answers on a graph of blood pressure against time.

51. A pigeon is released from a boat (point \( B \) in Figure 4.42) floating on a lake. Because of falling air over the cool water, the energy required to fly one meter over the lake is twice the corresponding energy \( e \) required for flying over the bank (\( e = 3 \) joule/meter). To minimize the energy required to fly from \( B \) to \( L \), the pigeon heads to a point \( P \) on the bank and then flies along the bank to \( L \). The distance \( AL \) is 2000 m, and \( AB \) is 500 m. The angle at \( A \) is a right angle.

(a) Express the energy required to fly from \( B \) to \( L \) via \( P \) as a function of the angle \( \theta \) (the angle \( BPA \)).

(b) What is the optimal angle \( \theta \)?

(c) Does your answer change if \( AL \), \( AB \), and \( e \) have different numerical values?

![Figure 4.42](image)

52. The bell-shaped curve of statistics has formula

\[
p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}
\]

where \( \mu \) is the mean and \( \sigma \) is the standard deviation.

(a) Where does \( p(x) \) have a maximum?

(b) Does \( p(x) \) have a point of inflection? If so, where?
The maximum revenue is achieved when the price is $65.

<table>
<thead>
<tr>
<th>Price, $p$</th>
<th>Number of trips sold, $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>300</td>
</tr>
<tr>
<td>75</td>
<td>330</td>
</tr>
<tr>
<td>70</td>
<td>360</td>
</tr>
<tr>
<td>65</td>
<td>390</td>
</tr>
</tbody>
</table>

Table 4.1 Demand for rafting trips

![Figure 4.48: Revenue for a rafting company as a function of price](image)

**Problems for Section 4.4**

1. Table 4.2 shows cost, $C(q)$, and revenue, $R(q)$.
   (a) At approximately what production level, $q$, is profit maximized? Explain your reasoning.
   (b) What is the price of the product?
   (c) What are the fixed costs?

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(q)$</td>
<td>0</td>
<td>1500</td>
<td>3000</td>
<td>4500</td>
<td>6000</td>
<td>7500</td>
<td>9000</td>
</tr>
<tr>
<td>$C(q)$</td>
<td>3000</td>
<td>3800</td>
<td>4200</td>
<td>4500</td>
<td>4800</td>
<td>5500</td>
<td>7400</td>
</tr>
</tbody>
</table>

Table 4.2

![Figure 4.50](image)

2. Figure 4.49 shows cost and revenue. For what production levels is the profit function positive? Negative? Estimate the production at which profit is maximized.

![Figure 4.49](image)

3. Using the cost and revenue graphs in Figure 4.50, sketch the following functions. Label the points $q_1$ and $q_2$.
   (a) Total profit
   (b) Marginal cost
   (c) Marginal revenue

![Figure 4.50](image)

4. A demand function is $p = 400 - 2q$, where $q$ is the quantity of the good sold for price $p$.
   (a) Find an expression for the total revenue, $R$, in terms of $q$.
   (b) Differentiate $R$ with respect to $q$ to find the marginal revenue, $MR$, in terms of $q$. Calculate the marginal revenue when $q = 10$.
   (c) Calculate the change in total revenue when production increases from $q = 10$ to $q = 11$ units. Confirm that a one unit increase in $q$ gives a reasonable approximation to the exact value of $MR$ obtained in part (b).

5. Let $C(q)$ represent the cost, $R(q)$ the revenue, and $\pi(q)$ the total profit, in dollars, of producing $q$ items.
   (a) If $C'(50) = 75$ and $R'(50) = 84$, approximately how much profit is earned by the 51st item?
   (b) If $C''(90) = 71$ and $R'(90) = 68$, approximately how much profit is earned by the 91st item?
   (c) If $\pi(q)$ is a maximum when $q = 78$, how do you think $C'(78)$ and $R'(78)$ compare? Explain.
6. Figure 4.46 on page 197 shows the points, $q_1$ and $q_2$, where marginal revenue equals marginal cost.

(a) On the graph of the corresponding total cost and total revenue functions in Figure 4.51, label the points $q_1$ and $q_2$. Using slopes, explain the significance of these points.

(b) Explain in terms of profit why one is a local minimum and one is a local maximum.

![Figure 4.51](image)

7. Table 4.3 shows marginal cost, $MC$, and marginal revenue, $MR$.

(a) Use the marginal cost and marginal revenue at a production of $q = 5000$ to determine whether production should be increased or decreased from 5000.

(b) Estimate the production level that maximizes profit.

Table 4.3

<table>
<thead>
<tr>
<th>$q$</th>
<th>5000</th>
<th>6000</th>
<th>7000</th>
<th>8000</th>
<th>9000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MR$</td>
<td>60</td>
<td>58</td>
<td>56</td>
<td>55</td>
<td>54</td>
<td>53</td>
</tr>
<tr>
<td>$MC$</td>
<td>48</td>
<td>52</td>
<td>54</td>
<td>55</td>
<td>58</td>
<td>63</td>
</tr>
</tbody>
</table>

8. Marginal revenue and marginal cost are given in the following table. Estimate the production levels that could maximize profit. Explain.

<table>
<thead>
<tr>
<th>$q$</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MR$</td>
<td>78</td>
<td>76</td>
<td>74</td>
<td>72</td>
<td>70</td>
<td>68</td>
</tr>
<tr>
<td>$MC$</td>
<td>100</td>
<td>80</td>
<td>70</td>
<td>65</td>
<td>75</td>
<td>90</td>
</tr>
</tbody>
</table>

9. Figure 4.52 shows cost and revenue for a product.

(a) Estimate the production level that maximizes profit.

(b) Graph marginal revenue and marginal cost for this product on the same coordinate system. Label on this graph the production level that maximizes profit.

![Figure 4.52](image)

10. Figure 4.53 shows graphs of marginal cost and marginal revenue. Estimate the production levels that could maximize profit. Explain your reasoning.

![Figure 4.53](image)

11. The marginal cost and marginal revenue of a company are $MC(q) = 0.03q^2 - 1.4q + 34$ and $MR(q) = 30$, where $q$ is the number of items manufactured. To increase profits, should the company increase or decrease production from each of the following levels?

(a) 25 items  
(b) 50 items  
(c) 80 items

12. A manufacturing process has marginal costs given in the table; the item sells for $30 per unit. At how many quantities, $q$, does the profit appear to be a maximum? In what intervals do these quantities appear to lie?

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MC$ ($/unit)$</td>
<td>34</td>
<td>23</td>
<td>18</td>
<td>19</td>
<td>26</td>
<td>39</td>
<td>58</td>
</tr>
</tbody>
</table>

13. Cost and revenue functions are given in Figure 4.54. Approximately what quantity maximizes profits?

![Figure 4.54](image)

14. Cost and revenue functions are given in Figure 4.54.

(a) At a production level of $q = 3000$, is marginal cost or marginal revenue greater? Explain what this tells you about whether production should be increased or decreased.

(b) Answer the same questions for $q = 5000$.

15. When production is 2000, marginal revenue is $4 per unit and marginal cost is $3.25 per unit. Do you expect maximum profit to occur at a production level above or below 2000? Explain.
16. Revenue is given by $R(q) = 450q$ and cost is given by $C(q) = 10,000 + 3q^2$. At what quantity is profit maximized? What is the total profit at this production level?

17. The demand equation for a product is $p = 45 - 0.01q$. Write the revenue as a function of $q$ and find the quantity that maximizes revenue. What price corresponds to this quantity? What is the total revenue at this price?

18. Revenue and cost functions for a company are given in Figure 4.55.
(a) Estimate the marginal cost at $q = 400$.
(b) Should the company produce the 500th item? Why?
(c) Estimate the quantity which maximizes profit.

![Figure 4.55](image)

19. The following table gives the cost and revenue, in dollars, for different production levels, $q$.

(a) At approximately what production level is profit maximized?
(b) What price is charged per unit for this product?
(c) What are the fixed costs of production?

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(q)$</td>
<td>0</td>
<td>1000</td>
<td>2000</td>
<td>3000</td>
<td>4000</td>
<td>5000</td>
</tr>
<tr>
<td>$C(q)$</td>
<td>700</td>
<td>900</td>
<td>1000</td>
<td>1100</td>
<td>1300</td>
<td>1900</td>
</tr>
</tbody>
</table>

20. An ice cream company finds that at a price of $4.00, demand is 4000 units. For every $0.25 decrease in price, demand increases by 200 units. Find the price and quantity sold that maximize revenue.

21. At a price of $8$ per ticket, a musical theater group can fill every seat in the theater, which has a capacity of 1500. For every additional dollar charged, the number of people buying tickets decreases by 75. What ticket price maximizes revenue?

22. The demand for tickets to an amusement park is given by $p = 70 - 0.02q$, where $p$ is the price of a ticket in dollars and $q$ is the number of people attending at that price.
(a) What price generates an attendance of 3000 people? What is the total revenue at that price? What is the total revenue if the price is $20? 
(b) Write the revenue function as a function of attendance, $q$, at the amusement park.
(c) What attendance maximizes revenue?
(d) What price should be charged to maximize revenue?
(e) What is the maximum revenue? Can we determine the corresponding profit?

23. The demand equation for a quantity $q$ of a product at price $p$, in dollars, is $p = -5q + 4000$. Companies producing the product report the cost, $C$, in dollars, to produce a quantity $q$ is $C = 6q + 5$ dollars.
(a) Express a company’s profit, in dollars, as a function of $q$.
(b) What production level earns the company the largest profit?
(c) What is the largest profit possible?

24. (a) Production of an item has fixed costs of $10,000 and variable costs of $2 per item. Express the cost, $C$, of producing $q$ items.
(b) The relationship between price, $p$, and quantity, $q$, demanded is linear. Market research shows that 10,100 items are sold when the price is $5 and 12,872 items are sold when the price is $4.50. Express $q$ as a function of $p$.
(c) Express the profit earned as a function of $q$.
(d) How many items should the company produce to maximize profit? (Give your answer to the nearest integer.) What is the profit at that production level?

25. You run a small furniture business. You sign a deal with a customer to deliver up to 400 chairs, the exact number to be determined by the customer later. The price will be $90$ per chair up to 300 chairs, and above 300, the price will be reduced by $0.25$ per chair (on the whole order) for every additional chair over 300 ordered. What are the largest and smallest revenues your company can make under this deal?

26. A warehouse selling cement has to decide how often and in what quantities to reorder. It is cheaper, on average, to place large orders, because this reduces the ordering cost per unit. On the other hand, larger orders mean higher storage costs. The warehouse always orders cement in the same quantity, $q$. The total weekly cost, $C$, of ordering and storage is given by

$$C = \frac{a}{q} + bq,$$

where $a$, $b$ are positive constants.

(a) Which of the terms, $a/q$ and $bq$, represents the ordering cost and which represents the storage cost?
(b) What value of $q$ gives the minimum total cost?

27. A business sells an item at a constant rate of $r$ units per month. It reorders in batches of $q$ units, at a cost of $a + bq$ dollars per order. Storage costs are $k$ dollars per item per month, and, on average, $q/k$ items are in storage, waiting to be sold. [Assume $r, a, b, k$ are positive constants.]

(a) How often does the business reorder?