The slope of the cost function is close to zero at \( q = 2 \), and is positive everywhere else, so the slope is smallest at \( q = 2 \). The marginal cost is smallest at a production level of 2000 units. Since \( C(2) \approx 10,000 \), the total cost to produce 2000 units is about $10,000.

**Example 4**

If the revenue and cost functions, \( R \) and \( C \), are given by the graphs in Figure 2.56, sketch graphs of the marginal revenue and marginal cost functions, \( MR \) and \( MC \).

![Graphs of R and C](image)

**Solution**

The revenue graph is a line through the origin, with equation

\[ R = pq \]

where \( p \) represents the constant price, so the slope is \( p \) and

\[ MR = R'(q) = p. \]

The total cost is increasing, so the marginal cost is always positive. For small \( q \) values, the graph of the cost function is concave down, so the marginal cost is decreasing. For larger \( q \), say \( q > 100 \), the graph of the cost function is concave up and the marginal cost is increasing. Thus, the marginal cost has a minimum at about \( q = 100 \). (See Figure 2.57.)

![Graphs of MR and MC](image)

**Problems for Section 2.5**

1. In Figure 2.58, estimate the marginal cost when the level of production is 10,000 units and interpret it.

![Graph of C(q)](image)

2. In Figure 2.59, estimate the marginal revenue when the level of production is 600 units and interpret it.

![Graph of R(q)](image)
3. The function $C(q)$ gives the cost in dollars to produce $q$ barrels of olive oil.
   (a) What are the units of marginal cost?
   (b) What is the practical meaning of the statement $MC = 3$ for $q = 100$?

4. It costs $4800 to produce 1295 items and it costs $4830 to produce 1305 items. What is the approximate marginal cost at a production level of 1300 items?

5. In Figure 2.60, is marginal cost greater at $q = 5$ or at $q = 30$? At $q = 20$ or at $q = 40$? Explain.

6. Figure 2.61 shows part of the graph of cost and revenue for a car manufacturer. Which is greater, marginal cost or marginal revenue, at $q_1$?

(a) $q_1$?

(b) $q_2$?

7. Let $C(q)$ represent the total cost of producing $q$ items. Suppose $C(15) = 2300$ and $C'(15) = 108$. Estimate the total cost of producing: (a) 16 items (b) 14 items.

8. To produce 1000 items, the total cost is $5000 and the marginal cost is $25 per item. Estimate the costs of producing 1001 items, 999 items, and 1100 items.

9. Let $C(q)$ represent the cost and $R(q)$ represent the revenue, in dollars, of producing $q$ items.
   (a) If $C(50) = 4300$ and $C'(50) = 24$, estimate $C(52)$.
   (b) If $C'(50) = 24$ and $R'(50) = 35$, approximately how much profit is earned by the 51st item?
   (c) If $C'(100) = 38$ and $R'(100) = 35$, should the company produce the 101st item? Why or why not?

10. Cost and revenue functions for a charter bus company are shown in Figure 2.62. Should the company add a 50th bus? How about a 90th? Explain your answers using marginal revenue and marginal cost.

11. For $q$ units of a product, a manufacturer's cost is $C(q)$ dollars and revenue is $R(q)$ dollars, with $C(500) = 7200$, $R(500) = 9400$, $MC(500) = 15$, and $MR(500) = 20$.
   (a) What is the profit or loss at $q = 500$?
   (b) If production is increased from 500 to 501 units, by approximately how much does profit change?

12. A company's cost of producing $q$ liters of a chemical is $C(q)$ dollars; this quantity can be sold for $R(q)$ dollars. Suppose $C(2000) = 5930$ and $R(2000) = 7780$.
   (a) What is the profit at a production level of 2000?
   (b) If $MC(2000) = 2.1$ and $MR(2000) = 2.5$, what is the approximate change in profit if $q$ is increased from 2000 to 2001? Should the company increase or decrease production from $q = 2000$?
   (c) If $MC(2000) = 4.77$ and $MR(2000) = 4.32$, should the company increase or decrease production from $q = 2000$?

13. An industrial production process costs $C(q)$ million dollars to produce $q$ million units; these units then sell for $R(q)$ million dollars. If $C(2.1) = 5.1$, $R(2.1) = 6.9$, $MC(2.1) = 0.6$, and $MR(2.1) = 0.7$, calculate
   (a) The profit earned by producing 2.1 million units
   (b) The change in revenue if production increases from 2.1 to 2.14 million units.
   (c) The change in revenue if production decreases from 2.1 to 2.05 million units.
   (d) The change in profit in parts (b) and (c).

14. The cost of recycling $q$ tons of paper is given in the following table. Estimate the marginal cost at $q = 2000$. Give units and interpret your answer in terms of cost. At approximately what production level does marginal cost appear smallest?

<table>
<thead>
<tr>
<th>$q$ (tons)</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
<th>3500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(q)$ (dollars)</td>
<td>2500</td>
<td>3200</td>
<td>3640</td>
<td>3825</td>
<td>3900</td>
<td>4400</td>
</tr>
</tbody>
</table>

15. Let $C(q)$ be the total cost of producing a quantity $q$ of a certain product. See Figure 2.63.
   (a) What is the meaning of $C(0)$?
(b) Describe in words how the marginal cost changes as the quantity produced increases.
(c) Explain the concavity of the graph (in terms of economics).
(d) Explain the economic significance (in terms of marginal cost) of the point at which the concavity changes.
(e) Do you expect the graph of \( C(q) \) to look like this for all types of products?

**CHAPTER SUMMARY**

- **Rate of change**
  Average, instantaneous

- **Estimating derivatives**
  Estimate derivatives from a graph, table of values, or formula.

- **Interpretation of derivatives**
  Rate of change, slope, using units, instantaneous velocity.

- **Marginality**
  Marginal cost and marginal revenue

- **Second derivative**
  Concavity

- **Derivatives and graphs**
  Understand relation between sign of \( f' \) and whether \( f \) is increasing or decreasing. Sketch graph of \( f' \) from graph of \( f \). Marginal analysis.

**REVIEW PROBLEMS FOR CHAPTER TWO**

1. For the function shown in Figure 2.64, at what labeled points is the slope of the graph positive? Negative? At which labeled point does the graph have the greatest (i.e., most positive) slope? The least slope (i.e., negative and with the largest magnitude)?

2. The function in Figure 2.65 has \( f(4) = 25 \) and \( f'(4) = 1.5 \). Find the coordinates of the points \( A, B, C \).

3. Estimate \( f'(2) \) for \( f(x) = 3^x \). Explain your reasoning.

4. In a time of \( t \) seconds, a particle moves a distance of \( s \) meters from its starting point, where \( s = 3t^2 \).
   (a) Find the average velocity between \( t = 1 \) and \( t = 1 + h \) if:
      (i) \( h = 0.1 \), (ii) \( h = 0.01 \), (iii) \( h = 0.001 \).
   (b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time \( t = 1 \).

5. The population of the world reached 1 billion in 1804, 2 billion in 1927, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987 and 6 billion in 1999. Find the average rate of change of the population of the world, in people per minute, during each of these intervals. [That is, from 1804 to 1927, 1927 to 1960, etc.]

6. In a time of \( t \) seconds, a particle moves a distance of \( s \) meters from its starting point, where \( s = \sin(2t) \).
   (a) Find the average velocity between \( t = 1 \) and \( t = 1 + h \) if:
      (i) \( h = 0.1 \), (ii) \( h = 0.01 \), (iii) \( h = 0.001 \).
   (b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time \( t = 1 \).
Solution

Graphical approach: Suppose we move along the curve from left to right. To the left of A, the slope is positive; it starts very positive and decreases until the curve reaches A, where the slope is 0. Between A and C the slope is negative. Between A and B the slope is decreasing (getting more negative); it is most negative at B. Between B and C the slope is negative but increasing; at C the slope is zero. From C to the right, the slope is positive and increasing. The graph of the derivative function is shown in Figure 3.9.

Algebraic approach: $f$ is a cubic that goes to $+\infty$ as $x \to +\infty$ so

$$f(x) = ax^3 + bx^2 + cx + d$$

with $a > 0$. Hence,

$$f'(x) = 3ax^2 + 2bx + c,$$

whose graph is a parabola opening upward, as in Figure 3.9.

Problems for Section 3.1

For Problems 1–36, find the derivative. Assume $a, b, c, k$ are constants.

1. $y = 5$
2. $y = 3x$
3. $y = x^{12}$
4. $y = x^{-12}$
5. $y = x^{4/3}$
6. $y = 8t^3$
7. $y = 3t^4 - 2t^2$
8. $y = 5x + 13$
9. $f(x) = \frac{1}{x^4}$
10. $f(q) = q^3 + 10$
11. $y = x^2 + 5x + 9$
12. $y = 6x^3 + 4x^2 - 2x$
13. $y = 3x^2 + 7x - 9$
14. $y = 8t^3 - 4t^2 + 12t - 3$
15. $y = 4.2q^2 - 0.5q + 11.27$
16. $y = -3x^4 - 4x^3 - 6x + 2$
17. $g(t) = \frac{1}{t^5}$
18. $f(z) = -\frac{1}{z^{6.1}}$
19. $y = \frac{1}{r^{7/2}}$
20. $y = \sqrt{x}$
21. $h(\theta) = \frac{1}{\sqrt{\theta}}$
22. $f(x) = \sqrt[3]{\frac{1}{x^3}}$
23. $y = 3t^5 - 5\sqrt{t} + \frac{7}{t}$
24. $y = z^2 + \frac{1}{2z}$
25. $y = 3t^2 + \frac{12}{\sqrt{t}} - \frac{1}{t^2}$
26. $h(t) = \frac{3}{t} + \frac{4}{t^2}$
27. $y = \sqrt{x}(x + 1)$
28. $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2})$
29. $f(x) = kx^2$
30. $y = ax^2 + bx + c$
31. $Q = aP^2 + bP^3$
32. $v = at^2 + \frac{b}{t^2}$
33. $P = a + b\sqrt{t}$
34. $V = \frac{4}{3}\pi r^3 b$
35. $w = 3ab^2q$
36. $h(x) = \frac{ax}{c} + b$

37. Let $f(t) = t^2 - 4t + 5$.
   (a) Find $f'(t)$.
   (b) Find $f'(1)$ and $f'(2)$.
   (c) Use a graph of $f(t)$ to check that your answers to part (b) are reasonable. Explain.

38. Let $f(x) = x^2 + 1$. Compute the derivatives $f'(0), f'(1), f'(2)$, and $f'(-1)$. Check your answers graphically.

39. Let $f(x) = x^2 + 3x - 5$. Find $f'(0), f'(3), f'(-2)$.

40. The height of a sand dune (in centimeters) is represented by $f(t) = 700 - 3t^2$, where $t$ is measured in years since 2005. Find $f(5)$ and $f'(5)$. Using units, explain what each means in terms of the sand dune.

41. Find the rate of change of a population of size $P(t) = t^3 + 4t^2 + 3t$ at time $t = 2$.

42. If $f(t) = 2t^3 - 4t^2 + 3t - 1$, find $f'(t)$ and $f''(t)$.

43. If $f(t) = t^4 - 3t^2 + 5t$, find $f'(t)$ and $f''(t)$.

44. Zebra mussels are freshwater shellfish that first appeared in the St. Lawrence River in the early 1980s and have spread throughout the Great Lakes. Suppose that $t$ months after they appeared in a small bay, the number of zebra mussels is given by $Z(t) = 300t^2$. How many zebra mussels are in the bay after four months? At what rate is the population growing at that time? Give units.

45. (a) Find the equation of the tangent line to $f(x) = x^3$ at the point where $x = 2$.
   (b) Graph the tangent line and the function on the same axes. If the tangent line is used to estimate values of the function, will the estimates be overestimates or underestimates?
46. Find the equation of the line tangent to the graph of \( f(t) = 6t - t^2 \) at \( t = 4 \). Sketch the graph of \( f(t) \) and the tangent line on the same axes.

47. Find the equation of the line tangent to the graph of \( f(x) = 2x^3 - 5x^2 + 3x - 5 \) at \( x = 1 \).

48. Suppose \( W \) is proportional to \( r^3 \). The derivative \( dW/dr \) is proportional to what power of \( r \)?

49. The cost to produce \( q \) items is \( C(q) = 1000 + 2q^2 \) dollars. Find the marginal cost of producing the 25th item. Interpret your answer in terms of costs.

50. The demand curve for a product is given by \( q = 300 - 3p \), where \( p \) is the price of the product and \( q \) is the quantity consumers will buy at that price.

(a) Write the revenue as a function of price.

(b) Find the marginal revenue when the price is $10, and interpret your answer in terms of revenue.

(c) For what prices is the marginal revenue positive? For what prices is it negative?

51. The yield, \( Y \), of an apple orchard (measured in bushels of apples per acre) is a function of the amount \( x \) of fertilizer in pounds used per acre. Suppose

\[ Y = f(x) = 320 + 140x - 10x^2. \]

(a) What is the yield if 5 pounds of fertilizer is used per acre?

(b) Find \( f'(5) \). Give units with your answer and interpret it in terms of apples and fertilizer.

(c) Given your answer to part (b), should more or less fertilizer be used? Explain.

52. The demand for a product is given, for \( p, q \geq 0 \), by

\[ p = f(q) = 50 - 0.03q^2. \]

(a) Find the \( p \)- and \( q \)-intercepts for this function and interpret them in terms of demand for this product.

(b) Find \( f(20) \) and give units with your answer. Explain what it tells you in terms of demand.

(c) Find \( f'(20) \) and give units with your answer. Explain what it tells you in terms of demand.

53. The cost (in dollars) of producing \( q \) items is given by \( C(q) = 0.08q^3 + 75q + 1000 \).

(a) Find the marginal cost function.

(b) Find \( C(50) \) and \( C'(50) \). Give units with your answers and explain what each is telling you about costs of production.

54. Let \( f(x) = x^2 - 4x + 8 \). For what \( x \)-values is \( f'(x) = 0 \)?

55. Let \( f(x) = x^3 - 6x^2 - 15x + 20 \). Find \( f'(x) \) and all values of \( x \) for which \( f'(x) = 0 \). Explain the relationship between these values of \( x \) and the graph of \( f(x) \).

56. The graph of \( y = x^3 - 9x^2 - 16x + 1 \) has a slope of 5 at two points. Find the coordinates of the points.

57. Show that for any power function \( f(x) = x^n \), we have \( f'(1) = n \).

58. If the demand curve is a line, we can write \( p = b + mq \), where \( p \) is the price of the product, \( q \) is the quantity sold at that price, and \( b \) and \( m \) are constants.

(a) Write the revenue as a function of quantity sold.

(b) Find the marginal revenue function.

59. A ball is dropped from the top of the Empire State building. The height, \( y \), of the ball above the ground (in feet) is given as a function of time, \( t \), (in seconds) by

\[ y = 1250 - 16t^2. \]

(a) Find the velocity of the ball at time \( t \). What is the sign of the velocity? Why is this to be expected?

(b) When does the ball hit the ground, and how fast is it going at that time? Give your answer in feet per second and in miles per hour (1 ft/sec = 15/22 mph).

3.2 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The Exponential Function

What do we expect the graph of the derivative of the exponential function \( f(x) = a^x \) to look like? The graph of an exponential function with \( a > 1 \) is shown in Figure 3.10. The function increases slowly for \( x < 0 \) and more rapidly for \( x > 0 \), so the values of \( f' \) are small for \( x < 0 \) and larger for \( x > 0 \). Since the function is increasing for all values of \( x \), the graph of the derivative must lie above the \( x \)-axis. In fact, the graph of \( f' \) resembles the graph of \( f \) itself. We will see how this observation holds for \( f(x) = 2^x \) and \( g(x) = 3^x \).
Problems for Section 3.2

Differentiate the functions in Problems 1–22. Assume that \( A, B, \) and \( C \) are constants.

1. \( f(x) = 2e^x + x^2 \)
2. \( P = 3t^3 + 2e^t \)
3. \( y = 5t^2 + 4e^t \)
4. \( f(x) = x^3 + 3^x \)
5. \( y = 2^x + \frac{2}{x^3} \)
6. \( y = 5 \cdot 5^t + 6 \cdot 6^t \)
7. \( f(x) = 2^x + 2 \cdot 3^x \)
8. \( y = 4 \cdot 10^x - x^3 \)
9. \( y = 3x - 2 \cdot 4^x \)
10. \( y = 5 \cdot 2^x - 5x + 4 \)
11. \( P(t) = 3000(1.02)^t \)
12. \( P(t) = 12.41(0.94)^t \)
13. \( P(t) = Ce^t \).
14. \( y = B + Ae^t \)
15. \( f(x) = Ae^x - Bx^2 + C \)
16. \( y = 10^x + \frac{10}{x} \)
17. \( R = 3 \ln q \)
18. \( D = 10 - \ln p \)
19. \( y = t^2 + 5 \ln t \)
20. \( R(q) = q^2 - 2 \ln q \)
21. \( y = x^2 + 4x - 3 \ln x \)
22. \( f(t) = Ae^t + B \ln t \)

23. For \( f(t) = 4 - 2e^t \), find \( f'(-1), f'(0), \) and \( f'(1) \). Graph \( f(t) \), and draw tangent lines at \( t = -1, t = 0, \) and \( t = 1 \). Do the slopes of the lines match the derivatives you found?

24. Find the equation of the tangent line to the graph of \( y = 3^x \) at \( x = 1 \). Check your work by sketching a graph of the function and the tangent line on the same axes.

25. (a) Find the slope of the graph of \( f(x) = 1 - e^x \) at the point where it crosses the \( x \)-axis.
(b) Find the equation of the tangent line to the curve at this point.

26. Worldwide production of solar power, in megawatts, can be modeled by \( P(t) = 1040(1.3)^t \), where \( t \) is years since 2000. Find \( P(0), f'(0), f(15), \) and \( f'(15) \). Give units and interpret your answers in terms of solar power.

27. During the 1990s, the population of Hungary was approximated by

\[
P = 10.8(0.994)^t,
\]

where \( P \) is in millions and \( t \) is in years since 1990. Assume the trend continues.

(a) What does this model predict for the population of Hungary in the year 2010?
(b) How fast (in people/year) does this model predict Hungary's population will be decreasing in 2010?

28. Certain pieces of antique furniture increased very rapidly in value in the 1990s and 2000s. For example, the value of a particular rocking chair is well approximated by

\[
V = 75(1.35)^t,
\]

where \( V \) is in dollars and \( t \) is the number of years since 1995. Find the rate, in dollars per year, at which the value is increasing.

29. With a yearly inflation rate of 5%, prices are given by

\[
P = P_0(1.05)^t,
\]

where \( P_0 \) is the price in dollars when \( t = 0 \) and \( t \) is time in years. Suppose \( P_0 = 1 \). How fast (in cents/year) are prices rising when \( t = 10 \)?

30. For the cost function \( C = 1000 + 300 \ln q \) (in dollars), find the cost and the marginal cost at a production level of 500. Interpret your answers in economic terms.

31. The Global 2000 Report gave the world's population, \( P \), as 4.1 billion in 1975 and growing at 2% annually.

(a) Give a formula for \( P \) in terms of time, \( t \), measured in years since 1975.
(b) Find \( \frac{dP}{dt} \) and \( \frac{dP}{dt} \bigg|_{t=10} \), and \( \frac{dP}{dt} \bigg|_{t=25} \). What do each of these represent in practical terms?

32. In 1990, the population of Mexico was about 84 million and growing at 2.6% annually, while the population of the US was about 250 million and growing at 0.7% annually. Which population was growing faster, if we measure growth rates in people/year? Explain your answer.

33. (a) Find the equation of the tangent line to \( y = \ln x \) at \( x = 1 \).
(b) Use it to calculate approximate values for \( \ln(1.1) \) and \( \ln(2) \).
(c) Using a graph, explain whether the approximate values are smaller or larger than the true values. Would the same result have held if you had used the tangent line to estimate \( \ln(0.9) \) and \( \ln(0.5) \)? Why?

34. Using the equation of the tangent line to the graph of \( e^x \) at \( x = 0 \), show that

\[
e^x \geq 1 + x
\]

for all values of \( x \). A sketch may be helpful.

35. Find the value of \( c \) in Figure 3.16, where the line \( t \) tangent to the graph of \( y = 2^x \) at \( (0, 1) \) intersects the \( x \)-axis.
3.3 THE CHAIN RULE

We now see how to differentiate composite functions such as \( f(t) = \ln(3t) \) and \( g(x) = e^{-x^2} \).

The Derivative of a Composition of Functions

Suppose \( y = f(z) \) with \( z = g(t) \) for some inside function \( g \) and outside function \( f \), where \( f \) and \( g \) are differentiable. A small change in \( t \), called \( \Delta t \), generates a small change in \( z \), called \( \Delta z \). In turn, \( \Delta z \) generates a small change in \( y \), called \( \Delta y \). Provided \( \Delta t \) and \( \Delta z \) are not zero, we can say

\[
\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta t}.
\]

Since the derivative \( \frac{dy}{dt} \) is the limit of the quotient \( \frac{\Delta y}{\Delta t} \) as \( \Delta t \) gets smaller and smaller, this suggests

The Chain Rule

If \( y = f(z) \) and \( z = g(t) \) are differentiable, then the derivative of \( y = f(g(t)) \) is given by

\[
\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}.
\]

In words, the derivative of a composite function is the derivative of the outside function times the derivative of the inside function:

\[
\frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t).
\]

The following example shows us how to interpret the chain rule in practical terms.

Example 1

The amount of gas, \( G \), in gallons, consumed by a car depends on the distance traveled, \( s \), in miles, and \( s \) depends on the time, \( t \), in hours. If 0.05 gallons of gas is consumed for each mile traveled, and the car is traveling at 30 miles/hr, how fast is gas being consumed? Give units.

Solution

We expect the rate of gas consumption to be in gallons/hr. We are told that

\[
\frac{dG}{ds} = 0.05 \text{ gallons/mile}
\]

\[
\frac{ds}{dt} = 30 \text{ miles/hr.}
\]
Example 8  Suppose $1000 is deposited into a bank account that pays 8% annual interest, compounded continuously.

(a) Find a formula $f(t)$ for the balance $t$ years after the initial deposit.
(b) Find $f(10)$ and $f'(10)$ and explain what your answers mean in terms of money.

Solution  
(a) The balance is $f(t) = 1000e^{0.08t}$.

(b) Substituting $t = 10$ gives 

$$f(10) = 1000e^{(0.08)(10)} = 2225.54.$$  

This means that the balance is $2225.54 after 10 years.

To find $f'(10)$, we compute 

$$f'(t) = 1000(0.08)e^{0.08t} = 80e^{0.08t}.$$  

Therefore, 

$$f'(10) = 80e^{(0.08)(10)} = 178.04.$$  

This means that after 10 years, the balance is growing at the rate of about $178 per year.

Problems for Section 3.3

Find the derivative of the functions in Problems 1–34.

1. $(4x^2 + 1)^7$
2. $f(x) = (x + 1)^{99}$
3. $R = (q^2 + 1)^4$
4. $w = (t^2 + 1)^{100}$
5. $w = (t^3 + 1)^{100}$
6. $w = (5r - 6)^3$
7. $y = \sqrt{3} + 1$
8. $f(t) = e^{3t}$
9. $y = 0.7t$
10. $y = e^{-4t}$
11. $P = e^{-0.2t}$
12. $P = 50e^{-0.6t}$
13. $P = 200e^{0.12t}$
14. $y = 12 - 3x^2 + 2e^{3x}$
15. $y = 25 + 4x^2 + e^{5x}$
16. $f(x) = 6e^{5x} + e^{-x^2}$
17. $y = 5e^{3t+1}$
18. $w = e^{-3t^2}$
19. $y = \ln(5t + 1)$
20. $y = \ln(5t + 1)$
21. $f(x) = \ln(1 - x)$
22. $f(t) = \ln(t^2 + 1)$
23. $f(x) = \ln(1 - e^{-x})$
24. $f(x) = \ln(e^x + 1)$
25. $f(t) = 5 \ln(5t + 1)$
26. $g(t) = \ln(4t + 9)$
27. $y = 5 + \ln(3t + 2)$
28. $Q = 100(t^2 + 5)^{0.5}$
29. $y = 5x + \ln(x + 2)$
30. $y = (5 + e^x)^2$
31. $P = (1 + \ln x)^{0.5}$
32. $\sqrt{e^x + 1}$
33. $f(x) = \sqrt{1 - x^2}$
34. $f(\theta) = (e^\theta + e^{-\theta})^{-1}$
35. Find the equation of the tangent line to $y = e^{-2t}$ at $t = 0$. Check by sketching the graphs of $y = e^{-2t}$ and the tangent line on the same axes.

36. Find the equation of the tangent line to $y = e^{-2t}$ at $t = 0$. Check by sketching the graphs of $y = e^{-2t}$ and the tangent line on the same axes.

37. Find the equation of the tangent line to $f(x) = 10e^{-0.2x}$ at $x = 4$.

38. A firm estimates that the total revenue, $R$, received from the sale of $q$ goods is given by 

$$R = \ln(1 + 1000e^2).$$

Calculate the marginal revenue when $q = 10$.

39. According to the US Census, the world population $P$, in billions, is approximately

$$P = 6.342e^{0.0111t},$$

where $t$ is in years since January 1, 2004. At what rate was the world’s population increasing on that date? Give your answer in millions of people per year.

40. The cost of producing a quantity, $q$, of a product is given by 

$$C(q) = 1000 + 30e^{0.05q}$$

dollars. Find the cost and the marginal cost when $q = 50$. Interpret these answers in economic terms.

41. The demand curve for a product is given by 

$$q = f(p) = 10,000e^{-0.25p},$$

where $q$ is the quantity sold and $p$ is the price of the product, in dollars. Find $f(2)$ and $f'(2)$. Explain in economic terms what information each of these answers gives you.

42. At a time $t$ hours after it was administered, the concentration of a drug in the body is 

$$f(t) = 27e^{-0.14t}$$

ng/ml. What is the concentration 4 hours after it was administered? At what rate is the concentration changing at that time?
43. The balance, $B$, in a bank account $t$ years after a deposit of $5000$ is given by $B = 5000e^{0.08t}$. At what rate is the balance in the account changing at $t = 5$ years? Use units to interpret your answer in financial terms.

44. With time, $t$, in minutes, the temperature, $H$, in degrees Celsius, of a bottle of water put in the refrigerator at $t = 0$ is given by

$$H = 4 + 16e^{-0.02t}.$$  

How fast is the water cooling initially? After 10 minutes? Give units.

45. A fish population is approximated by $P(t) = 10e^{0.6t}$, where $t$ is in months. Calculate and use units to explain what each of the following tells us about the population:

(a) $P(12)$

(b) $P'(12)$

46. The distance, $s$, of a moving body from a fixed point is given as a function of time by $s = 20e^{t/2}$. Find the velocity, $v$, of the body as a function of $t$.

47. If you invest $P$ dollars in a bank account at an annual interest rate of $r\%$, then after $t$ years you will have $B$ dollars, where

$$B = P \left(1 + \frac{r}{100}\right)^t.$$  

(a) Find $dB/dt$, assuming $P$ and $r$ are constant. In terms of money, what does $dB/dt$ represent?

(b) Find $dB/dr$, assuming $P$ and $t$ are constant. In terms of money, what does $dB/dr$ represent?

48. Some economists suggest that an extra year of education increases a person's wages, on average, by about 14%. Assume you could make $10 per hour with your current level of education and that inflation increases wages at a continuous rate of 3.5% per year.

(a) How much would you make per hour with four additional years of education?

(b) What is the difference between your wages in 20 years time with and without the additional four years of education?

(c) Is the difference you found in part (b) increasing with time? If so, at what rate? (Assume the number of additional years of education stays fixed at four.)

3.4 THE PRODUCT AND QUOTIENT RULES

This section shows how to find the derivatives of products and quotients of functions.

The Product Rule

Suppose we know the derivatives of $f(x)$ and $g(x)$ and want to calculate the derivative of the product, $f(x)g(x)$. We start by looking at an example. Let $f(x) = x$ and $g(x) = x^2$. Then

$$f(x)g(x) = x \cdot x^2 = x^3,$$

so the derivative of the product is $3x^2$. Notice that the derivative of the product is not equal to the product of the derivatives, since $f'(x) = 1$ and $g'(x) = 2x$, so $f'(x)g'(x) = (1)(2x) = 2x$. In general, we have the following rule, which is justified on page 172 in the Focus on Theory section.

\[ \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \]
(b) Differentiating using the quotient rule yields

\[
\frac{d}{dx} \left( \frac{1}{1+e^x} \right) = \frac{\frac{d}{dx}(1)}{(1+e^x)^2} \frac{1 + e^x - 0}{1+e^x} = \frac{-e^x}{(1+e^x)^2}.
\]

(c) The quotient rule gives

\[
\frac{d}{dx} \left( \frac{e^x}{x^2} \right) = \frac{\frac{d}{dx}(e^x) x^2 - e^x \frac{d}{dx}(x^2)}{(x^2)^2} = \frac{e^x x^2 - e^x (2x)}{x^4} = e^x \left( \frac{x^2 - 2x}{x^4} \right) = e^x \left( \frac{x - 2}{x^3} \right).
\]

### Problems for Section 3.4

1. If \( f(x) = (2x + 1)(3x - 2) \), find \( f'(x) \) two ways: by using the product rule and by multiplying out. Do you get the same result?

2. If \( f(x) = x^3(x^3 + 5) \), find \( f'(x) \) two ways: by using the product rule and by multiplying out before taking the derivative. Do you get the same result? Should you?

For Problems 3–33, find the derivative. Assume that \( a, b, c, \) and \( k \) are constants.

3. \( f(x) = xe^x \)

4. \( f(t) = te^{-2t} \)

5. \( y = 5xe^{x^2} \)

6. \( y = t^2(3t + 1)^3 \)

7. \( y = x \ln x \)

8. \( y = (t^2 + 3)e^t \)

9. \( z = (3t + 1)(5t + 2) \)

10. \( y = (t^3 - 7t^2 + 1)e^t \)

11. \( P = t^2 \ln t \)

12. \( f(t) = \frac{5}{t} + \frac{6}{t^2} \)

13. \( f(x) = \frac{x^2 + 3}{x} \)

14. \( R = 3qe^{-q} \)

15. \( y = te^{-t^2} \)

16. \( f(z) = \sqrt{ze^{-z}} \)

17. \( g(p) = p \ln(2p + 1) \)

18. \( f(t) = 5e^{5 - 2t} \)

19. \( f(w) = (5w^2 + 3)e^{w^2} \)

20. \( y = x^2 e^x \)

21. \( w = (t^3 + 5t)(t^2 - 7t + 2) \)

22. \( z = (te^{3t} + e^{5t})^6 \)

23. \( f(x) = \frac{x}{e^x} \)

24. \( w = \frac{3z}{1 + 2z} \)

25. \( z = \frac{1 - t}{1 + t} \)

26. \( y = \frac{e^x}{1 + e^x} \)

27. \( w = \frac{3y + y^2}{5 + y} \)

28. \( y = \frac{1 + z}{\ln z} \)

29. \( f(t) = ae^{bt} \)

30. \( f(x) = (ax^2 + b)^3 \)

31. \( f(x) = axe^{-bx} \)

32. \( f(x) = \frac{ax + b}{cx + k} \)

33. \( g(\alpha) = e^{\alpha e^{-2\alpha}} \)

34. If \( f(x) = (3x + 8)(2x - 5) \), find \( f'(x) \) and \( f''(x) \).

35. Find the equation of the tangent line to the graph of \( f(x) = x^2e^{-x} \) at \( x = 0 \). Check by graphing this function and the tangent line on the same axes.

36. Find the equation of the tangent line to the graph of \( f(x) = \frac{2x - 5}{x + 1} \) at the point at which \( x = 0 \).

37. If \( p \) is price in dollars and \( q \) is quantity, demand for a product is given by

\[
q = 5000e^{-0.08p}.
\]

(a) What quantity is sold at a price of $10?

(b) Find the derivative of demand with respect to price when the price is $10 and interpret your answer in terms of demand.

38. The demand for a product is given in Problem 37. Find the revenue and the derivative of revenue with respect to price at a price of $10. Interpret your answers in economic terms.

39. The quantity of a drug, \( Q \) mg, present in the body \( t \) hours after an injection of the drug is given as

\[
Q = f(t) = 100te^{-0.5t}.
\]

Find \( f(1) \), \( f'(1) \), \( f(5) \), and \( f'(5) \). Give units and interpret the answers.
40. The quantity demanded of a certain product, \( q \), is given in terms of \( p \), the price, by
\[
q = 1000e^{-0.02p}
\]
(a) Write revenue, \( R \), as a function of price.
(b) Find the rate of change of revenue with respect to price.
(c) Find the revenue and rate of change of revenue with respect to price when the price is $10. Interpret your answers in economic terms.

41. A drug concentration curve is given by \( C = f(t) = 20te^{-0.04t} \), with \( C \) in mg/ml and \( t \) in minutes.
(a) Graph \( C \) against \( t \). Is \( f'(15) \) positive or negative? Is \( f'(45) \) positive or negative? Explain.
(b) Find \( f'(30) \) and \( f'(30) \) analytically. Interpret them in terms of the concentration of the drug in the body.

42. If \( \frac{df}{dt}(tf(t)) = 1 + f(t) \), what is \( f'(t) \)?

43. Let \( h(x) = t(x)s(x) \) and \( p(x) = t(x)/s(x) \), where \( t(x) \) and \( s(x) \) are shown in Figure 3.20. Estimate:
(a) \( h'(1) \)  
(b) \( h'(0) \)  
(c) \( p'(0) \)

![Figure 3.20](image)

44. The quantity, \( q \), of a certain skateboard sold depends on the selling price, \( p \), in dollars, so we write \( q = f(p) \). You are given that \( f(140) = 15,000 \) and \( f'(140) = -100 \).
(a) What do \( f(140) = 15,000 \) and \( f'(140) = -100 \) tell you about the sales of skateboards?
(b) The total revenue, \( R \), earned by the sale of skateboards is given by \( R = pq \). Find \( \frac{dR}{dp} \) when \( p = 140 \).
(c) What is the sign of \( \frac{dR}{dp} \)? If the skateboards are currently selling for $140, what happens to revenue if the price is increased to $141?

45. The derivative \( f' \) gives the (absolute) rate of change of a quantity \( f \), and \( f'/f \) gives the relative rate of change of the quantity. In this problem, we show that the product rule is equivalent to an additive rule for relative rates of change. Assume \( h = f \cdot g \) with \( f \neq 0 \) and \( g \neq 0 \).
(a) Show that the additive rule
\[
\frac{f'}{f} + \frac{g'}{g} = H'
\]
implies the product rule, by multiplying through by \( h \) and using the fact that \( h = f \cdot g \).
(b) Show that the product rule implies the additive rule in part (a), by starting with the product rule and dividing through by \( h = f \cdot g \).

3.5 DERIVATIVES OF PERIODIC FUNCTIONS

Since the sine and cosine functions are periodic, their derivatives must be periodic also. (Why?) Let's look at the graph of \( f(x) = \sin x \) in Figure 3.21 and estimate the derivative function graphically.

![Figure 3.21: The sine function](image)

First we might ask ourselves where the derivative is zero. (At \( x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2 \), etc.) Then ask where the derivative is positive and where it is negative. (Positive for \( -\pi/2 < x < \pi/2 \); negative for \( \pi/2 < x < 3\pi/2 \), etc.) Since the largest positive slopes are at \( x = 0, 2\pi \), and so on, and the largest negative slopes are at \( x = \pi, 3\pi \), and so on, we get something like the graph in Figure 3.22.
Problems for Section 3.5

Differentiate the functions in Problems 1–20. Assume that $A$ and $B$ are constants.

1. $y = 5 \sin x$
2. $P = 3 + \cos t$
3. $y = t^2 + 5 \cos t$
4. $y = B + A \sin t$
5. $R(q) = q^2 - 2 \cos q$
6. $y = 5 \sin x - 5x + 4$
7. $f(x) = \sin(3x)$
8. $R = \sin(5t)$
9. $W = 4 \cos(t^2)$
10. $y = 2 \cos(5t)$
11. $y = \sin(x^2)$
12. $y = A \sin(Bt)$
13. $z = \cos(4\theta)$
14. $y = 6 \sin(2t) + \cos(4t)$
15. $f(x) = x^2 \cos x$
16. $f(x) = 2x \sin(3x)$
17. $f(\theta) = \theta^3 \cos \theta$
18. $z = \frac{e^{t^2} + t}{\sin(2t)}$
19. $f(t) = \frac{t^2}{\cos t}$
20. $f(\theta) = \frac{\sin \theta}{\theta}$

23. If $t$ is the number of months since June, the number of bird species, $N$, found in an Ohio forest oscillates approximately according to the formula

$$N = f(t) = 19 + 9 \cos \left(\frac{\pi}{6} t\right).$$

(a) Graph $f(t)$ for $0 \leq t \leq 24$ and describe what it shows. Use the graph to decide whether $f'(1)$ and $f'(10)$ are positive or negative.

(b) Find $f'(t)$.

(c) Find and interpret $f(1)$, $f'(1)$, $f(10)$, and $f'(10)$.

24. Is the graph of $y = \sin(x^2)$ increasing or decreasing when $x = 10$? Is it concave up or concave down?

25. A company’s monthly sales, $S(t)$, are seasonal and given as a function of time, $t$, in months, by

$$S(t) = 2000 + 600 \sin \left(\frac{\pi}{6} t\right).$$

(a) Graph $S(t)$ for $t = 0$ to $t = 12$. What is the maximum monthly sales? What is the minimum monthly sales? If $t = 0$ is January 1, when during the year are sales highest?

(b) Find $S(2)$ and $S'(2)$. Interpret in terms of sales.

26. On page 68 of Section 1.10 the depth, $y$, in feet, of water in Portland, Maine is given in terms of $t$, the number of hours since midnight, by

$$y = 4.9 + 4.4 \cos \left(\frac{\pi}{6} t\right).$$

(a) Find $dy/dt$. What does $dy/dt$ represent, in terms of water level?

(b) For $0 \leq t \leq 24$, when is $dy/dt$ zero? (Figure 1.101 on page 68 may be helpful.) Explain what it means (in terms of water level) for $dy/dt$ to be zero.

CHAPTER SUMMARY

- Derivatives of elementary functions
  - Powers, polynomials, exponential functions, logarithms, periodic functions
- Derivatives of sums, differences, constant multiples
- Chain rule
- Product and quotient rules
- Tangent line approximation
(c) At time $t = 1$ the investment momentarily stopped increasing in value, though it started increasing again immediately afterward. At $t = 4$, the value peaked and began to decline. At $t = 6$, it started increasing again.

Example 5

Find the critical point of the function $f(x) = x^2 + bx + c$. What is its graphical significance?

Solution

Since $f'(x) = 2x + b$, the critical point $x$ satisfies the equation: $2x + b = 0$. Thus, the critical point is at $x = -b/2$. The graph of $f$ is a parabola and the critical point is its vertex. See Figure 4.11.

![Critical point](image)

Figure 4.11: Critical point of the parabola $f(x) = x^2 + bx + c$. (Sketched with $b, c > 0$)

Problems for Section 4.1

In Problems 1–4, indicate all critical points of the function $f$. How many critical points are there? Identify each critical point as a local maximum, a local minimum, or neither.

1. $f(t)$
2. $f(x)$
3. $f(x)$
4. $f(x)$

5. During an illness a person ran a fever. His temperature rose steadily for eighteen hours, then went steadily down for twenty hours. When was there a critical point for his temperature as a function of time?

6. (a) Graph a function with two local minima and one local maximum.
   (b) Graph a function with two critical points. One of these critical points should be a local minimum, and the other should be neither a local maximum nor a local minimum.

7. Graph two continuous functions $f$ and $g$, each of which has exactly five critical points, the points $A$–$E$ in Figure 4.12, and which satisfy the following conditions:
   (a) $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$
   (b) $g(x) \to -\infty$ as $x \to -\infty$ and $g(x) \to 0$ as $x \to \infty$

![Figure 4.12](image)

Using a calculator or computer, graph the functions in Problems 8–13. Describe briefly in words the interesting features of the graph including the location of the critical points and where the function is increasing/decreasing. Then use the derivative and algebra to explain the shape of the graph.

8. $f(x) = x^3 - 6x + 1$
9. $f(x) = x^3 + 6x + 1$
10. $f(x) = 3x^5 - 5x^3$
11. $f(x) = e^x - 10x$
12. $f(x) = x \ln x, \ x > 0$
13. $f(x) = x + 2 \sin x$