27. The 2004 US presidential debates questioned whether the minimum wage has kept pace with inflation. Decide the question using the following information: In 1938, the minimum wage was 25¢; in 2004, it was $5.15. During the same period, inflation averaged 4.3%.

28. Whooping cough was thought to have been almost wiped out by vaccinations. It is now known that the vaccination wears off, leading to an increase in the number of cases, \( w \), from 1248 in 1981 to 18,957 in 2004.

(a) With \( t \) in years since 1980, find an exponential function that fits this data.

(b) What does your answer to part (a) give as the average annual percent growth rate of the number of cases?

(c) On May 4, 2005, the Arizona Daily Star reported (correctly) that the number of cases had more than doubled between 2000 and 2004. Does your model confirm this report? Explain.

1.6 THE NATURAL LOGARITHM

In Section 1.5, we projected the population of McAllen, Texas (in thousands), by the function

\[ P = f(t) = 570(1.037)^t, \]

where \( t \) is the number of years since 2000. Now suppose that instead of calculating the population at time \( t \), we ask when the population will reach 900,000. We want to find the value of \( t \) for which

\[ 900 = f(t) = 570(1.037)^t. \]

We use logarithms to solve for a variable in an exponent.

Definition and Properties of the Natural Logarithm

We define the natural logarithm of \( x \), written \( \ln x \), as follows:

The natural logarithm of \( x \), written \( \ln x \), is the power of \( e \) needed to get \( x \). In other words,

\[ \ln x = c \quad \text{means} \quad e^c = x. \]

The natural logarithm is sometimes written \( \log_e x \).

For example, \( \ln e^3 = 3 \) since 3 is the power of \( e \) needed to give \( e^3 \). Similarly, \( \ln(1/e) = \ln e^{-1} = -1 \). A calculator gives \( \ln 5 = 1.6094 \), because \( e^{1.6094} = 5 \). However if we try to find \( \ln(-7) \) on a calculator, we get an error message because \( e \) to any power is never negative or 0. In general

\[ \ln x \text{ is not defined if } x \text{ is negative or 0.} \]

To work with logarithms, we use the following properties:

<table>
<thead>
<tr>
<th>Properties of the Natural Logarithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \ln(AB) = \ln A + \ln B )</td>
</tr>
<tr>
<td>2. ( \ln \left(\frac{A}{B}\right) = \ln A - \ln B )</td>
</tr>
<tr>
<td>3. ( \ln(A^p) = p \ln A )</td>
</tr>
<tr>
<td>4. ( \ln e^x = x )</td>
</tr>
<tr>
<td>5. ( e^{\ln x} = x )</td>
</tr>
</tbody>
</table>

In addition, \( \ln 1 = 0 \) because \( e^0 = 1 \), and \( \ln e = 1 \) because \( e^1 = e \).
Example 5  Sketch graphs of $P = e^{0.5t}$, a continuous growth rate of 50%, and $Q = 5e^{-0.2t}$, a continuous decay rate of 20%.

Solution  The graph of $P = e^{0.5t}$ is in Figure 1.65. Notice that the graph is the same shape as the previous exponential growth curves: increasing and concave up. The graph of $Q = 5e^{-0.2t}$ is in Figure 1.66; it has the same shape as other exponential decay functions.

![Graph of $P = e^{0.5t}$](image1)

![Graph of $Q = 5e^{-0.2t}$](image2)

Figure 1.65: Continuous exponential growth function

Figure 1.66: Continuous exponential decay function

Problems for Section 1.6

For Problems 1–16, solve for $t$ using natural logarithms.

1. $5^t = 7$
2. $10 = 2^t$
3. $2 = (1.02)^t$
4. $130 = 10^t$
5. $50 = 10 \cdot 3^t$
6. $100 = 25(1.5)^t$
7. $a = b^t$
8. $10 = e^t$
9. $5 = 2e^t$
10. $e^{3t} = 100$
11. $10 = 6e^{0.5t}$
12. $40 = 100e^{-0.03t}$
13. $B = Pe^{rt}$
14. $2P = Pe^{0.3t}$
15. $5e^{3t} = 8e^{2t}$
16. $7 \cdot 3^t = 5 \cdot 2^t$

The functions in Problems 17–20 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

17. $P = 5(1.07)^t$
18. $P = 7.7(0.92)^t$
19. $P = 15e^{-0.06t}$
20. $P = 3.2e^{0.03t}$

21. A city's population is 1000 and growing at 5% a year.
(a) Find a formula for the population at time $t$ years from now assuming that the 5% per year is an:
   (i) Annual rate  (ii) Continuous annual rate
(b) In each case in part (a), estimate the population of the city in 10 years.

22. The following formulas give the populations of four different towns, $A$, $B$, $C$, and $D$, with $t$ in years from now.

- $P_A = 6000e^{0.08t}$
- $P_B = 1000e^{-0.02t}$
- $P_C = 12000e^{0.03t}$
- $P_D = 9000e^{0.12t}$

(a) Which town is growing fastest (that is, has the largest percentage growth rate)?
(b) Which town is the largest now?
(c) Are any of the towns decreasing in size? If so, which one(s)?

Write the functions in Problems 23–26 in the form $P = P_0a^t$.
Which represent exponential growth and which represent exponential decay?

23. $P = 15e^{0.25t}$
24. $P = 2e^{-0.5t}$
25. $P = P_0e^{0.21}$
26. $P = 7e^{-\pi t}$

In Problems 27–30, put the functions in the form $P = P_0e^{kt}$.

27. $P = 15(1.5)^t$
28. $P = 10(1.7)^t$
29. $P = 174(0.9)^t$
30. $P = 4(0.55)^t$
31. A fishery stocks a pond with 1000 young trout. The number of trout \( t \) years later is given by \( P(t) = 1000e^{-0.5t} \).

(a) How many trout are left after six months? After 1 year?
(b) Find \( P(3) \) and interpret it in terms of trout.
(c) At what time are there 100 trout left?
(d) Graph the number of trout against time, and describe how the population is changing. What might be causing this?

32. During a recession a firm's revenue declines continuously so that the revenue, \( R \) (measured in millions of dollars), in \( t \) years is given by \( R = 5e^{-0.15t} \).

(a) Calculate the current revenue and the revenue in two years' time.
(b) After how many years will the revenue decline to $2.7 million?

33. (a) What is the continuous percent growth rate for \( P = 100e^{0.004t} \), with time, \( t \), in years?
(b) Write this function in the form \( P = P_0 e^t \). What is the annual percent growth rate?

34. (a) What is the annual percent decay rate for \( P = 25(0.88)^t \), with time, \( t \), in years?
(b) Write this function in the form \( P = P_0 e^{kt} \). What is the continuous percent decay rate?

35. The gross world product is \( W = 32.4(1.036)^t \), where \( W \) is in trillions of dollars and \( t \) is years since 2001. Find a formula for gross world product using a continuous growth rate.

36. The population, \( P \), in millions, of Nicaragua was 5.4 million in 2004 and growing at an annual rate of 3.4%. Let \( t \) be time in years since 2004.

(a) Express \( P \) as a function in the form \( P = P_0 a^t \).
(b) Express \( P \) as an exponential function using base \( e \).
(c) Compare the annual and continuous growth rates.

37. What annual percent growth rate is equivalent to a continuous percent growth rate of 8%?

38. What continuous percent growth rate is equivalent to an annual percent growth rate of 10%?

39. The population of the world can be represented by \( P = 6.4(1.0126)^t \), where \( P \) is in billions of people and \( t \) is years since 2004. Find a formula for the population of the world using a continuous growth rate.

40. The population of a city is 50,000 in 2001 and is growing at a continuous yearly rate of 4.5%.

(a) Give the population of the city as a function of the number of years since 2001. Sketch a graph of the population against time.
(b) What will be the population of the city in the year 2011?
(c) Calculate the time for the population of the city to reach 100,000. This is called the doubling time of the population.

41. In 1980, there were about 170 million vehicles (cars and trucks) and about 227 million people in the United States. The number of vehicles has been growing at 4% a year, while the population has been growing at 1% a year. When was there, on average, one vehicle per person?

1.7 EXPONENTIAL GROWTH AND DECAY

Many quantities in nature change according to an exponential growth or decay function of the form \( P = P_0 e^{kt} \), where \( P_0 \) is the initial quantity and \( k \) is the continuous growth or decay rate.

**Example 1**

The Environmental Protection Agency (EPA) recently investigated a spill of radioactive iodine. The radiation level at the site was about 2.4 millirems/hour (four times the maximum acceptable limit of 0.6 millirems/hour), so the EPA ordered an evacuation of the surrounding area. The level of radiation from an iodine source decays at a continuous hourly rate of \( k = -0.004 \).

(a) What was the level of radiation 24 hours later?
(b) Find the number of hours until the level of radiation reached the maximum acceptable limit, and the inhabitants could return.

**Solution**

(a) The level of radiation, \( R \), in millirems/hour, at time \( t \), in hours since the initial measurement, is given by

\[
R = 2.4e^{-0.004t},
\]

so the level of radiation 24 hours later was

\[
R = 2.4e^{(-0.004)(24)} = 2.18 \text{ millirems per hour}.
\]
Problems for Section 1.7

1. The half-life of nicotine in the blood is 2 hours. A person absorbs 0.4 mg of nicotine by smoking a cigarette. Fill in the following table with the amount of nicotine remaining in the blood after $t$ hours. Estimate the length of time until the amount of nicotine is reduced to 0.04 mg.

<table>
<thead>
<tr>
<th>$t$ (hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nicotine (mg)</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. If you deposit $10,000 in an account earning interest at an 8% annual rate compounded continuously, how much money is in the account after five years?

3. If you need $20,000 in your bank account in 6 years, how much must be deposited now? The interest rate is 10%, compounded continuously.

4. If a bank pays 6% per year interest compounded continuously, how long does it take for the balance in an account to double?

5. You invest $5000 in an account which pays interest compounded continuously.
   (a) How much money is in the account after 8 years, if the annual interest rate is 4%?
   (b) If you want the account to contain $8000 after 8 years, what yearly interest rate is needed?

6. Suppose $1000 is invested in an account paying interest at a rate of 5.5% per year. How much is in the account after 8 years if the interest is compounded
   (a) Annually?
   (b) Continuously?

7. Find the doubling time of a quantity that is increasing by 7% per year.

8. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.
   (a) Write a formula for the amount, $A$ mg, of caffeine in the body $t$ hours after drinking a cup of coffee.
   (b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
   (c) Use logarithms to find the half-life of caffeine.

9. A population, currently 200, is growing at 5% per year.
   (a) Write a formula for the population, $P$, as a function of time, $t$, years in the future.
   (b) Graph $P$ against $t$.
   (c) Estimate the population 10 years from now.
   (d) Use the graph to estimate the doubling time of the population.

10. The antidepressant fluoxetine (or Prozac) has a half-life of about 3 days. What percentage of a dose remains in the body after one day? After one week?

11. You need $10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

12. A firm decides to increase output at a constant rate from its current level of 20,000 to 30,000 units during the next five years. Calculate the annual percent rate of increase required to achieve this growth.

13. The quantity, $Q$, of radioactive carbon-14 remaining $t$ years after an organism dies is given by the formula

   \[ Q = Q_0 e^{-0.00121t}, \]

   where $Q_0$ is the initial quantity.
   (a) A skull uncovered at an archeological dig has 15% of the original amount of carbon-14 present. Estimate its age.
   (b) Calculate the half-life of carbon-14.

14. (a) Figure 1.68 shows exponential growth. Starting at $t = 0$, estimate the time for the population to double.
   (b) Repeat part (a), but this time start at $t = 3$.
   (c) Pick any other value of $t$ for the starting point, and notice that the doubling time is the same no matter where you start.

15. An exponentially growing animal population numbers 500 at time $t = 0$; two years later, it is 1500. Find a formula for the size of the population in $t$ years and find the size of the population at $t = 5$.

16. If the quantity of a substance decreases by 4% in 10 hours, find its half-life.

17. Figure 1.69 shows the balances in two bank accounts. Both accounts pay the same interest rate, but one compounds continuously and the other compounds annually. Which curve corresponds to which compounding method? What is the initial deposit in each case?
18. Pregnant women metabolize some drugs at a slower rate than the rest of the population. The half-life of caffeine is about 4 hours for most people. In pregnant women, it is 10 hours.\textsuperscript{44} (This is important because caffeine, like all psychoactive drugs, crosses the placenta to the fetus.) If a pregnant woman and her husband each have a cup of coffee containing 100 mg of caffeine at 8 am, how much caffeine does each have left in the body at 10 pm?\textsuperscript{19}

19. The half-life of radioactive strontium-90 is 29 years. In 1960, radioactive strontium-90 was released into the atmosphere during testing of nuclear weapons, and was absorbed into people’s bones. How many years does it take until only 10\% of the original amount absorbed remains?\textsuperscript{20}

20. If $12,000 is deposited in an account paying 8\% interest per year, compounded continuously, how long will it take for the balance to reach $20,000?\textsuperscript{21}

21. You want to invest money for your child’s education in a certificate of deposit (CD). You want it to be worth $12,000 in 10 years. How much should you invest if the CD pays interest at a 9\% annual rate compounded (a) Annually? (b) Continuously?\textsuperscript{22}

22. When you rent an apartment, you are often required to give the landlord a security deposit which is returned if you leave the apartment undamaged. In Massachusetts the landlord is required to pay the tenant interest on the deposit once a year, at a 5\% annual rate, compounded annually. The landlord, however, may invest the money at a higher (or lower) interest rate. Suppose the landlord invests a $1000 deposit at a yearly rate of (a) 6\%, compounded continuously (b) 4\%, compounded continuously.

In each case, determine the net gain or loss by the landlord at the end of the first year. (Give your answer to the nearest cent.)\textsuperscript{23}

23. In 1923, koalas were introduced on Kangaroo Island off the coast of Australia. In 1996, the population was 5000. By 2005, the population had grown to 27,000, prompting a debate on how to control their growth and avoid koalas dying of starvation.\textsuperscript{45} Assuming exponential growth, find the (continuous) rate of growth of the koala population between 1996 and 2005. Find a formula for the population as a function of the number of years since 1996, and estimate the population in the year 2020.\textsuperscript{24}

24. The total world marine catch in 1950 was 17 million tons and in 2001 was 99 million tons.\textsuperscript{46} If the marine catch is increasing exponentially, find the (continuous) rate of increase. Use it to predict the total world marine catch in the year 2020.\textsuperscript{25}

25. (a) Use the Rule of 70 to predict the doubling time of an investment which is earning 8\% interest per year. (b) Find the doubling time exactly, and compare your answer to part (a).\textsuperscript{26}

26. The island of Manhattan was sold for $24 in 1626. Suppose the money had been invested in an account which compounded interest continuously.

(a) How much money would be in the account in the year 2005 if the yearly interest rate was (i) 5\%? (ii) 7\%?

(b) If the yearly interest rate was 6\%, in what year would the account be worth one million dollars?\textsuperscript{27}

27. Owing to an innovative rural public health program, infant mortality in Senegal, West Africa, is being reduced at a rate of 10\% per year. How long will it take for infant mortality to be reduced by 50\%?\textsuperscript{28}

28. In 2004, the world’s population was 6.4 billion, and the population was projected to reach 8.5 billion by the year 2030. What annual growth rate is projected?\textsuperscript{29}

29. A picture supposedly painted by Vermeer (1632–1675) contains 99.5\% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.\textsuperscript{30}

30. A business associate who owes you $3000 offers to pay you $2800 now, or else pay you three yearly installments of $1000 each, with the first installment paid now. If you use only financial reasons to make your decision, which option should you choose? Justify your answer, assuming a 6\% interest rate per year, compounded continuously.\textsuperscript{31}

31. Big Tree McGee is negotiating his rookie contract with a professional basketball team. They have agreed to a three-year deal which will pay Big Tree a fixed amount at the end of each of the three years, plus a signing bonus at the beginning of his first year. They are still haggling about the amounts and Big Tree must decide between a big signing bonus and fixed payments per year, or a smaller bonus with payments increasing each year. The two options are summarized in the table. All values are payments in millions of dollars.

<table>
<thead>
<tr>
<th></th>
<th>Signing bonus</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option #1</td>
<td>6.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Option #2</td>
<td>1.0</td>
<td>2.0</td>
<td>4.0</td>
<td>6.0</td>
</tr>
</tbody>
</table>

(a) Big Tree decides to invest all income in stock funds which he expects to grow at a rate of 10\% per year, compounded continuously. He would like to choose the contract option which gives him the greater future value at the end of the three years when the last payment is made. Which option should he choose? (b) Calculate the present value of each contract offer.


\textsuperscript{45}News.yahoo.com/s/afp/australiananimalskoalas, accessed June 1, 2005.

\textsuperscript{46}The \textit{World Almanac and Book of Facts} 2005, p. 143 (New York).
32. A company is considering whether to buy a new machine, which costs $97,000. The cash flows (adjusted for taxes and depreciation) that would be generated by the new machine are given in the following table:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>$50,000</td>
<td>$40,000</td>
<td>$25,000</td>
<td>$20,000</td>
</tr>
</tbody>
</table>

(a) Find the total present value of the cash flows. Treat each year's cash flow as a lump sum at the end of the year and use an interest rate of 7.5% per year, compounded annually.

(b) Based on a comparison of the cost of the machine and the present value of the cash flows, would you recommend purchasing the machine?

33. You win $38,000 in the state lottery to be paid in two installments—$19,000 now and $19,000 one year from now. A friend offers you $36,000 in return for your two lottery payments. Instead of accepting your friend's offer, you take out a one-year loan at an interest rate of 8.25% per year, compounded annually. The loan will be paid back by a single payment of $19,000 (your second lottery check) at the end of the year. Which is better, your friend's offer or the loan?

34. You are considering whether to buy or lease a machine whose purchase price is $12,000. Taxes on the machine will be $580 due in one year, $464 due in two years, and $290 due in three years. If you buy the machine, you expect to be able to sell it after three years for $5,000. If you lease the machine for three years, you make an initial payment of $2650 and then three payments of $2650 at the end of each of the next three years. The leasing company will pay the taxes. The interest rate is 7.75% per year, compounded annually. Should you buy or lease the machine? Explain.

35. You are buying a car that comes with a one-year warranty and are considering whether to purchase an extended warranty for $375. The extended warranty covers the two years immediately after the one-year warranty expires. You estimate that the yearly expenses that would have been covered by the extended warranty are $150 at the end of the first year of the extension and $250 at the end of the second year of the extension. The interest rate is 5% per year, compounded annually. Should you buy the extended warranty? Explain.

36. You have the option of renewing the service contract on your three-year-old dishwasher. The new service contract is for three years at a price of $200. The interest rate is 7.25% per year, compounded annually, and you estimate that the costs of repairs if you do not buy the service contract will be $50 at the end of the first year, $100 at the end of the second year, and $150 at the end of the third year. Should you buy the service contract? Explain.

### 1.8 NEW FUNCTIONS FROM OLD

We have studied linear and exponential functions, and the logarithm function. In this section, we learn how to create new functions by composing, stretching, and shifting functions we already know.

#### Composite Functions

A drop of water falls onto a paper towel. The area, \( A \) of the circular damp spot is a function of \( r \), its radius, which is a function of time, \( t \). We know \( A = f(r) = \pi r^2 \); suppose \( r = g(t) = t + 1 \). By substitution, we express \( A \) as a function of \( t \):

\[
A = f(g(t)) = \pi(t + 1)^2.
\]

The function \( f(g(t)) \) is a "function of a function," or a **composite function**, in which there is an **inside function** and an **outside function**. To find \( f(g(2)) \), we first add one (\( g(2) = 2 + 1 = 3 \)) and then square and multiply by \( \pi \). We have

\[
f(g(2)) = \pi(2 + 1)^2 = \pi \cdot 3^2 = 9\pi.
\]

The inside function is \( t + 1 \) and the outside function is squaring and multiplying by \( \pi \). In general, the inside function represents the calculation that is done first and the outside function represents the calculation done second.

**Example 1**

If \( f(t) = t^2 \) and \( g(t) = t + 2 \), find

(a) \( f(t + 1) \)  
(b) \( f(t) + 3 \)  
(c) \( f(t + h) \)  
(d) \( f(g(t)) \)  
(e) \( g(f(t)) \)
(b) To see the effect of the new factory, look at an example. At a price of 10 dollars, approximately 800 units are currently produced. With the new factory, this amount increases by 100 units, so the new amount produced is 900 units. At each price, the quantity produced increases by 100, so the new supply curve is \( S \) shifted horizontally to the right by 100 units. (See Figure 1.77.)

![Figure 1.76: New cost function (original curve dashed)](image1)

![Figure 1.77: New supply curve (original curve dashed)](image2)

### Problems for Section 1.8

1. For \( g(x) = x^2 + 2x + 3 \), find and simplify:
   
   (a) \( g(2 + h) \)
   (b) \( g(2) \)
   (c) \( g(2 + h) - g(2) \)

2. If \( f(x) = x^2 + 1 \), find and simplify:
   
   (a) \( f(t + 1) \)
   (b) \( f(t^2 + 1) \)
   (c) \( f(2) \)
   (d) \( 2f(t) \)
   (e) \( f(t)^2 + 1 \)

For the functions \( f \) and \( g \) in Problems 3–6, find

(a) \( f(g(1)) \)
(b) \( g(f(1)) \)
(c) \( f(g(x)) \)
(d) \( g(f(x)) \)
(e) \( f(g(t)) \)

3. \( f(x) = x^2 \), \( g(x) = x + 1 \)
4. \( f(x) = \sqrt{x + 4}, g(x) = x^2 \)
5. \( f(x) = e^x, g(x) = x^2 \)
6. \( f(x) = 1/x, g(x) = 3x + 4 \)
7. Let \( f(x) = x^2 \) and \( g(x) = 3x - 1 \). Find the following:
   
   (a) \( f(2) + g(2) \)
   (b) \( f(2) \cdot g(2) \)
   (c) \( f(g(2)) \)
   (d) \( g(f(2)) \)

In Problems 8–10, find the following

(a) \( f(g(x)) \)
(b) \( g(f(x)) \)
(c) \( f(f(x)) \)
8. \( f(x) = 2x^2 \) and \( g(x) = x + 3 \)
9. \( f(x) = 2x + 3 \) and \( g(x) = 5x^2 \)
10. \( f(x) = x^2 + 1 \) and \( g(x) = \ln x \)

11. Use the variable \( u \) for the inside function to express each of the following as a composite function:
   
   (a) \( y = 2^{3u-1} \)
   (b) \( P = \sqrt{5t^2 + 10} \)
   (c) \( w = 2 \ln(3r + 4) \)

12. Use the variable \( u \) for the inside function to express each of the following as a composite function:
   
   (a) \( y = (5t^2 - 2)^6 \)
   (b) \( P = 12e^{-0.6t} \)
   (c) \( C = 12\ln{(q^3 + 1)} \)

In Problems 13–16, use Figure 1.78 to graph the functions.

![Figure 1.78](image3)

13. \( n(t) = m(t) + 2 \)
14. \( p(t) = m(t - 1) \)
15. \( k(t) = m(t + 1.5) \)
16. \( w(t) = m(t - 0.5) - 2.5 \)

Graph the functions in Problems 17–22 using Figure 1.79.

![Figure 1.79](image4)

17. \( y = f(x) + 2 \)
18. \( y = 2f(x) \)
19. \( y = f(x - 1) \)
20. \( y = -3f(x) \)
21. \( y = 2f(x) - 1 \)
22. \( y = 2 - f(x) \)
31. Make a table of values for each of the following functions using Table 1.28.
   (a) \( f(x) + 3 \)  \hspace{1cm} (b) \( f(x - 2) \)  \hspace{1cm} (c) \( 5g(x) \)
   (d) \( -f(x) + 2 \)  \hspace{1cm} (e) \( g(x - 3) \)  \hspace{1cm} (f) \( f(x) + g(x) \)

For Problems 32–34, use the graphs in Figure 1.81.

32. Estimate \( f(g(1)) \).
33. Estimate \( g(f(2)) \).
34. Estimate \( f(f(1)) \).

35. (a) Write an equation for a graph obtained by vertically stretching the graph of \( y = x^2 \) by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
   (b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
   (c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

36. Using Table 1.29, create a table of values for \( f(g(x)) \) and for \( g(f(x)) \).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
f(x) & 0 & 1 & 2 & 3 & 2 & 1 & 0 \\
g(x) & 3 & 2 & 2 & 0 & -2 & -2 & -3 \\
\hline
\end{array}
\]

37. A plan is adopted to reduce the pollution in a lake to the legal limit. The quantity \( Q \) of pollutants in the lake after \( t \) weeks of clean-up is modeled by the function \( Q = f(t) \) where \( f(t) = A + Be^{Ct} \).
   (a) What are the signs of \( A, B \) and \( C \)?
   (b) What is the initial quantity of pollution in the lake?
   (c) What is the legal limit of pollution in the lake?

1.9 PROPORTIONALITY, POWER FUNCTIONS, AND POLYNOMIALS

Proportionality

A common functional relationship occurs when one quantity is proportional to another. For example, if apples are $1.40 a pound, we say the price you pay, \( p \) dollars, is proportional to the weight you buy, \( w \) pounds, because

\[ p = f(w) = 1.40w. \]

As another example, the area, \( A \), of a circle is proportional to the square of the radius, \( r \):

\[ A = f(r) = \pi r^2. \]
From Figure 1.88, we see that the two graphs look indistinguishable. The reason is that the leading term of each polynomial (the one with the highest power of \( x \)) is the same, namely \( x^4 \), and for the large values of \( x \) in this window, the leading term dominates the other terms.

Figure 1.88: Graphs of \( y = x^4 \) and \( y = x^4 - 15x^2 - 15x \) look almost indistinguishable in a large window.

Problem 41 compares the graphs of these two functions in a smaller window with Figure 1.88.

We see in Example 7 that, from a distance, the polynomial \( y = x^4 - 15x^2 - 15x \) looks like the power function \( y = x^4 \). In general, if the graph of a polynomial of degree \( n \)

\[
y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]

is viewed in a large enough window, it has approximately the same shape as the graph of the power function given by the leading term:

\[
y = a_n x^n.
\]

Problems for Section 1.9

In Problems 1–12, determine whether or not the function is a power function. If it is a power function, write it in the form \( y = k x^p \) and give the values of \( k \) and \( p \).

1. \( y = 5\sqrt{x} \)  
2. \( y = \frac{3}{x^2} \)  
3. \( y = 2^x \)  
4. \( y = \frac{3}{8x} \)  
5. \( y = (3x^5)^2 \)  
6. \( y = \frac{5}{2\sqrt{x}} \)  
7. \( y = 3 \cdot 5^x \)  
8. \( y = \frac{2x^2}{10} \)  
9. \( y = \frac{8}{x} \)  
10. \( y = (5x)^3 \)  
11. \( y = 3x^2 + 4 \)  
12. \( y = \frac{x}{5} \)

In Problems 13–16, write a formula representing the function.

13. The strength, \( S \), of a beam is proportional to the square of its thickness, \( h \).
14. The energy, \( E \), expended by a swimming dolphin is proportional to the cube of the speed, \( v \), of the dolphin.
15. The average velocity, \( v \), for a trip over a fixed distance, \( d \), is inversely proportional to the time of travel, \( t \).
16. The gravitational force, \( F \), between two bodies is inversely proportional to the square of the distance \( d \) between them.
17. The specific heat, \( s \), of an element is the number of calories of heat required to raise the temperature of one gram of the element by one degree Celsius. Use the following table to decide if \( s \) is proportional or inversely proportional to the atomic weight, \( w \), of the element. If so, find the constant of proportionality.

<table>
<thead>
<tr>
<th>Element</th>
<th>Li</th>
<th>Mg</th>
<th>Al</th>
<th>Fe</th>
<th>Ag</th>
<th>Pb</th>
<th>Hg</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w )</td>
<td>6.9</td>
<td>24.3</td>
<td>27.0</td>
<td>55.8</td>
<td>107.9</td>
<td>207.2</td>
<td>200.6</td>
</tr>
<tr>
<td>( s )</td>
<td>0.92</td>
<td>0.25</td>
<td>0.21</td>
<td>0.11</td>
<td>0.056</td>
<td>0.031</td>
<td>0.033</td>
</tr>
</tbody>
</table>

18. The following table gives values for a function \( p = f(t) \). Could \( p \) be proportional to \( t \)?

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>0</td>
<td>25</td>
<td>60</td>
<td>100</td>
<td>140</td>
<td>200</td>
</tr>
</tbody>
</table>
19. The blood mass of a mammal is proportional to its body mass. A rhinoceros with body mass 3000 kilograms has blood mass of 150 kilograms. Find a formula for the blood mass of a mammal as a function of the body mass and estimate the blood mass of a human with body mass 70 kilograms.

20. The number of species of lizards, $N$, found on an island off Baja California is proportional to the fourth root of the area, $A$, of the island.\(^49\) Write a formula for $N$ as a function of $A$. Graph this function. Is it increasing or decreasing? Is the graph concave up or concave down? What does this tell you about lizards and island area?

21. The surface area of a mammal, $S$, satisfies the equation $S = kM^{2/3}$, where $M$ is the body mass, and the constant of proportionality $k$ depends on the body shape of the mammal. A human of body mass 70 kilograms has surface area 18,600 cm\(^2\). Find the constant of proportionality for humans. Find the surface area of a human with body mass 60 kilograms.

22. Biologists estimate that the number of animal species of a certain body length is inversely proportional to the square of the body length.\(^50\) Write a formula for the number of animal species, $N$, of a certain body length as a function of the length, $L$. Are there more species at large lengths or at small lengths? Explain.

23. The circulation time of a mammal (that is, the average time it takes for all the blood in the body to circulate once and return to the heart) is proportional to the fourth root of the body mass of the mammal.

(a) Write a formula for the circulation time, $T$, in terms of the body mass $B$.

(b) If an elephant of body mass 5230 kilograms has a circulation time of 148 seconds, find the constant of proportionality.

(c) What is the circulation time of a human with body mass 70 kilograms?

24. Allometry is the study of the relative size of different parts of a body as a consequence of growth. In this problem, you will check the accuracy of an allometric equation: the weight of a fish is proportional to the cube of its length.\(^51\) Table 1.30 relates the weight, $y$, in gm, of plaice (a type of fish) to its length, $x$, in cm. Does this data support the hypothesis that (approximately) $y = kx^3$? If so, estimate the constant of proportionality, $k$.

25. The DuBois formula relates a person’s surface area, $s$, in m\(^2\), to weight $w$, in kg, and height $h$, in cm, by $s = 0.01w^{0.29}h^{0.75}$.

(a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?

(b) What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m\(^2\)?

(c) For people of fixed weight 70 kg, solve for $h$ as a function of $s$. Simplify your answer.

26. According to the National Association of Realtors,\(^52\) the minimum annual gross income, $m$, in thousands of dollars, needed to obtain a 30-year home loan of $A$ thousand dollars at 9% is given in Table 1.31. Note that the larger the loan, the greater the income needed.

27. Of course, not every mortgage is financed at 9%. In fact, excepting for slight variations, mortgage interest rates are generally determined not by individual banks but by the economy as a whole. The minimum annual gross income, $m$, in thousands of dollars, needed for a home loan of $100,000 at various interest rates, $r$, is given in Table 1.32. Note that obtaining a loan at a time when interest rates are high requires a larger income.

(a) Is the size of the loan, $A$, proportional to the minimum annual gross income, $m$?

(b) Is the percentage rate, $r$, proportional to the minimum annual gross income, $m$?

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\(^{50}\) US News & World Report, August 18, 1997, p. 79.


27. A standard tone of 20,000 dynes/cm² (about the loudness of a rock band) is assigned a value of 10. A subject listened to other sounds, such as a light whisper, normal conversation, thunder, a jet plane at takeoff, and so on. In each case, the subject was asked to judge the loudness and assign it a number relative to 10, the value of the standard tone. This is a “judgment of magnitude” experiment. The power law \( J = a l^{0.3} \) was found to model the situation well, where \( l \) is the actual loudness (measured in dynes/cm²) and \( J \) is the judged loudness.

(a) What is the value of \( a \)?
(b) What is the judged loudness if the actual loudness is 0.2 dynes/cm² (normal conversation)?
(c) What is the actual loudness if judged loudness is 20?

For the functions in Problems 28–35:
(a) What is the degree of the polynomial? Is the leading coefficient positive or negative?
(b) What power function approximates \( f(x) \) for large \( x \)? Without using a calculator or computer, sketch the graph of the function in a large window.
(c) Using a calculator or computer, sketch a graph of the function. How many turning points does the function have? How does the number of turning points compare to the degree of the polynomial?

28. \( f(x) = 5x^3 - 17x^2 + 9x + 50 \)
29. \( f(x) = x^2 + 10x - 5 \)
30. \( f(x) = 8x - 3x^2 \)
31. \( f(x) = 17 + 8x - 2x^3 \)
32. \( f(x) = -9x^5 + 82x^3 + 12x^2 \)
33. \( f(x) = 100 + 5x - 12x^2 + 3x^3 - x^4 \)
34. \( f(x) = 0.01x^5 + 2.3x^2 - 7 \)
35. \( f(x) = 0.2x^4 + 1.5x^3 - 3x^3 + 9x - 15 \)

36. Each of the graphs in Figure 1.89 is of a polynomial. The windows are large enough to show global behavior.
(a) What is the minimum possible degree of the polynomial?
(b) Is the leading coefficient of the polynomial positive or negative?

37. A sporting goods wholesaler finds that when the price of a product is $25, the company sells 600 units per week. When the price is $30, the number sold per week decreases to 460 units.

(a) Find the demand, \( q \), as a function of price, \( p \), assuming that the demand curve is linear.
(b) Use your answer to part (a) to write revenue as a function of price.
(c) Graph the revenue function in part (b). Find the price that maximizes revenue. What is the revenue at this price?

38. A health club has cost and revenue functions given by \( C = 10,000 + 35q \) and \( R = pq \), where \( q \) is the number of annual club members and \( p \) is the price of a one-year membership. The demand function for the club is \( q = 3000 - 20p \).

(a) Use the demand function to write cost and revenue as functions of \( p \).
(b) Graph cost and revenue as a function of \( p \), on the same axes. (Note that price does not go above $170 and that the annual costs of running the club reach $120,000.)
(c) Explain why the graph of the revenue function has the shape it does.
(d) For what prices does the club make a profit?
(e) Estimate the annual membership fee that maximizes profit. Mark this point on your graph.

39. Use shifts of power functions to find a possible formula for each of the graphs:

40. Find a calculator window in which the graphs of \( f(x) = x^3 + 1000x^2 + 1000 \) and \( g(x) = x^3 - 1000x^2 - 1000 \) appear indistinguishable.

41. Do the functions \( y = x^4 \) and \( y = x^4 - 15x^2 - 15x \) look similar in the window \(-4 \leq x \leq 4; -100 \leq y \leq 100\)? Comment on the difference between your answer to this question and what you see in Figure 1.88.
Example 7

On April 25, 2005, high tide in Portland, Maine was at midnight. The height of the water in the harbor is a periodic function, since it oscillates between high and low tide. If \( t \) is in hours since midnight, the height (in feet) is approximated by the formula

\[
y = 4.9 + 4.4 \cos \left( \frac{\pi}{6} t \right).
\]

(a) Graph this function from \( t = 0 \) to \( t = 24 \).
(b) What was the water level at high tide?
(c) When was low tide, and what was the water level at that time?
(d) What is the period of this function, and what does it represent in terms of tides?
(e) What is the amplitude of this function, and what does it represent in terms of tides?

Solution

(a) See Figure 1.101.
(b) The water level at high tide was 9.3 feet (given by the \( y \)-intercept on the graph).
(c) Low tide occurs at \( t = 6 \) (6 am) and at \( t = 18 \) (6 pm). The water level at this time is 0.5 feet.
(d) The period is 12 hours and represents the interval between successive high tides or successive low tides. Of course, there is something wrong with the assumption in the model that the period is 12 hours. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as it in fact does. The interval between successive high tides actually averages about 12 hours and 39 minutes, which could be taken into account in a more precise mathematical model.
(e) The maximum is 9.3, and the minimum is 0.5, so the amplitude is \((9.3 - 0.5)/2\), which is 4.4 feet. This represents half the difference between the depths at high and low tide.

\[
\begin{array}{c|c}
\text{y (feet)} & y = 4.9 + 4.4 \cos \left( \frac{\pi}{6} t \right) \\
9.3 & \text{Oscillation} \\
4.9 & \\
0.5 & \\
12 \text{ mid.} & 6 \text{ am} & 12 \text{ noon} & 6 \text{ pm} & 12 \text{ mid.}
\end{array}
\]

Figure 1.101: Graph of the function approximating the depth of the water in Portland, Maine on April 25, 2005

Problems for Section 1.10

1. A graduate student in environmental science studied the temperature fluctuations of a river. Figure 1.102 shows the temperature of the river (in °C) every hour, with hour 0 being midnight of the first day.

(a) Explain why a periodic function could be used to model these data.
(b) Approximately when does the maximum occur? The minimum? Why does this make sense?
(c) What is the period for these data? What is the amplitude?

\[
\begin{array}{c|c}
\text{°C} & \\
32 & \\
31 & \\
30 & \\
29 & \\
28 & \\
14 24 48 72 96 108 & \text{hours}
\end{array}
\]

Figure 1.102

---

For Problems 19–30, find a possible formula for each graph.

21. \( y \)

22. \( y \)

23. \( y \)

24. \( y \)

25. \( y \)

26. \( y \)

27. \( y \)

28. \( y \)

29. \( y \)

30. \( y \)

31. The depth of water in a tank oscillates once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

32. The desert temperature, \( H \), oscillates daily between 40° F at 5 am and 80° F at 5 pm. Write a possible formula for \( H \) in terms of \( t \), measured in hours from 5 am.

33. Table 1.33 gives values for \( g(t) \), a periodic function.
   (a) Estimate the period and amplitude for this function.
   (b) Estimate \( g(34) \) and \( g(60) \).

34. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, \( y \) meters, is given as a function of time, \( t \), in hours since midnight by

\[
y = D + A \cos(B(t - C)).
\]

(a) What is the physical meaning of \( D \)?
(b) What is the value of \( A \)?
(c) What is the value of \( B \)? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of \( C \)?

### Chapter Summary

- **Function terminology**
  Domain/range, increasing/decreasing, concavity, intercepts.

- **Linear functions**
  Slope, \( y \)-intercept. Grow by equal amounts in equal times.

- **Economic applications**

- **Change, average rate of change, relative rate of change**

- **Exponential functions**
  Exponential growth and decay, growth rate, the number \( e \), continuous growth rate, doubling time, half life, compound interest. Grow by equal percentages in equal times.

- **The natural logarithm function**

- **New functions from old**
  Composition, shifting, stretching.

- **Power functions and proportionality**

- **Polynomials**

- **Periodic functions**
  Sine, cosine, amplitude, period.
Find the period and amplitude in Problems 2–4.

2. \( y = 7 \sin(3t) \)
3. \( z = 3 \cos(u/4) + 5 \)
4. \( r = 0.1 \sin(\pi t) + 2 \)

5. A person breathes in and out every three seconds. The volume of air in the person's lungs varies between a minimum of 2 liters and a maximum of 4 liters. Which of the following is the best formula for the volume of air in the person's lungs as a function of time?

(a) \( y = 2 + 2 \sin\left(\frac{\pi}{3} t\right) \)
(b) \( y = 3 + \sin\left(\frac{2\pi}{3} t\right) \)
(c) \( y = 2 + 2 \sin\left(\frac{2\pi}{3} t\right) \)
(d) \( y = 3 + \sin\left(\frac{\pi}{3} t\right) \)

6. Sketch a possible graph of sales of sunscreen in the northeastern US over a 3-year period, as a function of months since January 1 of the first year. Explain why your graph should be periodic. What is the period?

For Problems 7–12, sketch graphs of the functions. What are their amplitudes and periods?

7. \( y = 3 \sin x \)
8. \( y = 3 \sin 2x \)
9. \( y = -3 \sin 2\theta \)
10. \( y = 4 \cos 2x \)
11. \( y = 4 \cos\left(\frac{1}{2} t\right) \)
12. \( y = 5 - \sin 2t \)

13. Delta Cephei is one of the most visible stars in the night sky. Its brightness has periods of 5.4 days, the average brightness is 4.0 and its brightness varies by \( \pm 0.35 \). Find a formula that models the brightness of Delta Cephei as a function of time, \( t \), with \( t = 0 \) at peak brightness.

14. Values of a function are given in the following table. Explain why this function appears to be periodic. Approximately what are the period and amplitude of the function? Assuming that the function is periodic, estimate its value at \( t = 15 \), at \( t = 75 \), and at \( t = 135 \).

\[
\begin{array}{c|ccccccccc}
 t & 20 & 25 & 30 & 35 & 40 & 45 & 50 & 55 & 60 \\
 f(t) & 1.8 & 1.4 & 1.7 & 2.3 & 2.0 & 1.8 & 1.4 & 1.7 & 2.3 \\
\end{array}
\]

15. The following table shows values of a periodic function \( f(x) \). The maximum value attained by the function is 5.

(a) What is the amplitude of this function?
(b) What is the period of this function?
(c) Find a formula for this periodic function.

\[
\begin{array}{c|cccccccc}
 x & 0 & 2 & 4 & 6 & 8 & 10 & 12 \\
 f(x) & 5 & 0 & -5 & 0 & 5 & 0 & -5 \\
\end{array}
\]

16. Figure 1.103 shows the levels of the hormones estrogen and progesterone during the monthly ovarian cycles in females.\(^{56}\) Is the level of both hormones periodic? What is the period in each case? Approximately when in the monthly cycle is estrogen at a peak? Approximately when in the monthly cycle is progesterone at a peak?

![Figure 1.103](image)

17. Figure 1.104 shows the number of reported\(^{57}\) cases of mumps by month, in the US, for 1972–73.

(a) Find the period and amplitude of this function, and interpret each in terms of mumps.
(b) Predict the number of cases of mumps 30 months and 45 months after January 1, 1972.

![Figure 1.104](image)

18. Most breeding birds in the northeast US migrate elsewhere during the winter. The number of bird species in an Ohio forest preserve oscillates between a high of 28 in June and a low of 10 in December.\(^{58}\)

(a) Graph the number of bird species in this preserve as a function of \( t \), the number of months since June. Include at least three years on your graph.
(b) What are the amplitude and period of this function?
(c) Find a formula for the number of bird species, \( B \), as a function of the number of months, \( t \) since June.

---


\(^{57}\) Center for Disease Control, 1974, *Reported Morbidity and Mortality in the United States 1973*, Vol. 22, No. 53. Prior to the licensing of the vaccine in 1967, 100,000–200,000 cases of mumps were reported annually. Since 1995, fewer than 1000 cases are reported annually. Source: CDC.