Assignment 9

Chapter 4: 3, 5-9

4.3. Let \( f \) be a nonnegative measurable function. Show that \( \int f = 0 \) implies \( f = 0 \) a.e.

Proof.

Let \( f \) be a nonnegative measurable function. Let \( E := \{ x : f(x) \neq 0 \} \). Because \( f \) is nonnegative \( E = \{ x : f(x) > 0 \} \). For each \( n \in \mathbb{Z}^+ \) define \( E_n := \{ x : f(x) > \frac{1}{n} \} \). Because \( f \) is measurable, \( E \) and each \( E_n \) are measurable. It is easy to check that \( E = \bigcup_{n=1}^{\infty} E_n \).

Let \( n \in \mathbb{Z}^+ \). Clearly \( E_n \subseteq E \). By Lemma 1 below \( \int_E f \geq \int_{E_n} f \). By Proposition 4.8(iii) (p. 85, Royden) \( \int_{E_n} f \geq \int_{E_n} \frac{1}{n} \). Hence

\[
0 = \int_E f \geq \int_{E_n} \frac{1}{n} = \int \frac{1}{n} \cdot \chi_{E_n}.
\]

Because \( E_n \) is measurable, \( \frac{1}{n} \cdot \chi_{E_n} \) is a simple function. By definition \( \int \frac{1}{n} \cdot \chi_{E_n} = \frac{1}{n} \cdot m(E_n) \). In particular this means that

\[
0 \geq \frac{1}{n} \cdot m(E_n) \geq 0.
\]

In other words \( m(E_n) = 0 \) for all \( n \in \mathbb{Z}^+ \). By Proposition 3.13 (p. 62, Royden)

\[
m(E) \leq \sum_{n=1}^{\infty} m(E_n) = 0.
\]

This establishes that \( m(\{ x : f(x) \neq 0 \}) = 0 \) as desired.

\[\blacksquare\]

Lemma 1 Let \( f \) be a nonnegative measurable function and \( E, F \subseteq \mathbb{R} \) such that \( E, F \) are both measurable and \( E \subseteq F \). Then \( \int_F f \leq \int_E f \).

Proof.
Let \( f \) be a nonnegative measurable function and \( E, F \subseteq \mathbb{R} \) such that \( E, F \) are both measurable and \( E \subseteq F \). It follows from Problem 3.20. (p. 70, Royden) that \( \chi_E = \chi_F + \chi_{E \setminus F} \). Hence
\[
(\chi_E \cdot f)(x) = (\chi_F \cdot f)(x) + (\chi_{E \setminus F} \cdot f)(x) \geq (\chi_F \cdot f)(x).
\]
for all \( x \in \mathbb{R} \), so \( \chi_E \cdot f \geq \chi_F \cdot f \). By Proposition 3.13 (p. 62, Royden)
\[
\int_E f = \int \chi_E \cdot f \geq \int \chi_F \cdot f = \int_F f.
\]
4.5. Let $f$ be a nonnegative integrable function. Show that the function $F$ defined by

$$F(x) = \int_{-\infty}^{x} f$$

is continuous by using Theorem 10.

**Proof.**

Let $f$ be a nonnegative integrable function. For each $n \in \mathbb{Z}^+$ define the function $f_n$ by the rule

$$f_n(x) = \left(f \cdot \chi_{(-\infty,a-\frac{1}{n}]}(x)\right)$$

Now $\chi_{(-\infty,a-\frac{1}{n}]}$ is measurable because $(-\infty,a-\frac{1}{n}]$ is measurable, and $f$ is measurable by definition. By Proposition 3.19 (p. 67, Royden) $f_n$ is measurable. It is easy to check that $f_n \rightarrow f \cdot \chi_{(-\infty,a]}$ pointwise except at $a$. It follows immediately that $f_n \rightarrow f \cdot \chi_{(-\infty,a]}$ pointwise almost everywhere. By the Monotone Convergence Theorem (p. 87, Royden)

$$F(a) = \lim_{n \rightarrow \infty} \int_{-\infty}^{a} f \cdot \chi_{(-\infty,a-\frac{1}{n}]} = \lim_{n \rightarrow \infty} \int_{a-\frac{1}{n}}^{a} f$$

Now we'll show that $F(a) = \lim_{n \rightarrow \infty} \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f$ by demonstrating that $\lim_{n \rightarrow \infty} \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f = 0$. Let $\varepsilon > 0$. By Proposition 4.14 (p. 88, Royden) choose $\eta$ such that for every set $A \subseteq \mathbb{R}$ with $m(A) < \eta$ we have $\int_{A} f < \varepsilon$. Choose $N \in \mathbb{Z}^+$ such that $\frac{1}{N} < \frac{\eta}{2}$. Let $n \geq N$. Then

$$m\left(a - \frac{1}{N}, a + \frac{1}{N}\right) = 2 \cdot \frac{1}{N} < 2 \cdot \frac{\eta}{2} = \eta$$

which implies that $\int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f < \varepsilon$. Hence $\lim_{n \rightarrow \infty} \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f = 0$. By Proposition 4.12 (p. 87, Royden)

$$F(a) = \lim_{n \rightarrow \infty} \int_{-\infty}^{a-\frac{1}{n}} f + 0 = \lim_{n \rightarrow \infty} \int_{-\infty}^{a-\frac{1}{n}} f + \lim_{n \rightarrow \infty} \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f$$

$$= \lim_{n \rightarrow \infty} \left(\int_{-\infty}^{a-\frac{1}{n}} f + \int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f\right) = \lim_{n \rightarrow \infty} \int_{-\infty}^{a+\frac{1}{n}} f.$$ 

This establishes that

$$F(a) = \lim_{n \rightarrow \infty} \int_{-\infty}^{a-\frac{1}{n}} f = \lim_{n \rightarrow \infty} \int_{-\infty}^{a+\frac{1}{n}} f.$$
Now we’ll show that \( F(a) = \lim_{x \to a} F(x) \). Let \( \varepsilon > 0 \) and choose \( M_1, M_2 \in \mathbb{Z}^+ \) such that \( n_1 \geq M_1 \) and \( n_2 \geq M_2 \) implies

\[
\left| \int_{-\infty}^{a} f - \int_{-\infty}^{a - \frac{1}{n_1}} f \right| < \varepsilon \text{ and } \left| \int_{-\infty}^{a} f - \int_{-\infty}^{a + \frac{1}{n_2}} f \right| < \varepsilon.
\]

Let \( \delta = \frac{1}{\max(M_1, M_2)} \). Let \( x \in \mathbb{R} \) and suppose that \( 0 < |x - a| < \delta \). If \( x < a \), it follows immediately that \( a - \delta < x < a \). By Proposition 4.12 (p. 87, Royden)

\[
\int_{-\infty}^{a} f = \int_{-\infty}^{x} f + \int_{x}^{a} f \geq \int_{-\infty}^{x} f.
\]

As \( a - \frac{1}{M_1} < a - \delta < x \), by similar reasoning \( \int_{-\infty}^{x} f \geq \int_{-\infty}^{a - \frac{1}{M_1}} f \). Hence

\[
\left| \int_{-\infty}^{a} f - \int_{-\infty}^{x} f \right| = \int_{-\infty}^{a} f - \int_{-\infty}^{x} f \leq \int_{-\infty}^{a} f - \int_{-\infty}^{a - \frac{1}{M_1}} f
\]

\[
= \left| \int_{-\infty}^{a} f - \int_{-\infty}^{a - \frac{1}{M_1}} f \right| < \varepsilon.
\]

If \( x > a \), an analogous argument gives the same result. This covers all cases for \( x \) and establishes that \( F(a) = \lim_{x \to a} F(x) \). Equivalently, \( F \) is continuous at \( a \).

As our choice of \( a \) was arbitrary, we have \( F \) continuous at every point \( a \in \mathbb{R} \). Hence \( F \) is continuous.
4.6. Let \( \{f_n\} \) be a sequence of nonnegative measurable functions that converge to \( f \), and suppose \( f_n \leq f \) for each \( n \). Then

\[
\int f = \lim_{n \to \infty} \int f_n.
\]

Proof.

Let \( \{f_n\} \) be a sequence of nonnegative measurable functions that converge to \( f \), and suppose \( f_n \leq f \) for each \( n \).

By Fatou’s Lemma we have \( \int f \leq \liminf \int f_n \).

Now we’ll show that \( \int f \geq \limsup \int f_n \). Let \( m \in \mathbb{Z}^+ \). By hypothesis \( \int f_k \leq \int f \) for all \( k \in \mathbb{Z}^+ \). This holds in particular for all \( k \geq m \), so

\[
\sup_{k \geq m} \int f_k \leq \int f.
\]

By definition

\[
\liminf \int f_n = \inf_n \sup_{k \geq n} \int f_k \leq \sup_{k \geq m} \int f_k \leq \int f.
\]

This establishes that

\[
\liminf \int f_n \leq \int f \leq \limsup \int f_n.
\]

Because it is always true that \( \liminf \int f_n \geq \lim \int f_n \), we have

\[
\liminf \int f_n = \int f = \lim \int f_n.
\]

In other words \( \int f = \lim \int f_n \).

\[\blacksquare\]
4.7.  (a) Show that we may have strict inequality in Fatou’s Lemma.

(b) Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions.

(a) Show that we may have strict inequality in Fatou’s Lemma.

Proof.

Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function. Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \chi_{[n, n+1)}(x).$$

Note that $f_n$ is a simple function and $[n, n+1)$ is measurable by Theorem 3.12. (p. 61, Royden) because $[n, n+1) = [n, \infty) \cap (-\infty, n+1)$ is a Borel set. By definition

$$\int f_n = m([n, n+1)) = 1$$

for all $n \in \mathbb{Z}^+$. 

Now we’ll show that $f_n(x) \to f$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Choose $N \in \mathbb{Z}^+$ such that $N > x$. Then for all $n \geq N$, we have $x \notin [n, n+1)$. In other words

$$f_n(x) = 0 < \varepsilon.$$

This establishes that $f_n \to f$.

Hence

$$\int f = \int 0 = 0 < 1 = \lim_{n \to \infty} \int f_n$$

demonstrating that we may have strict inequality in Fatou’s Lemma.

(b) Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions.

Proof.

Again let $f : \mathbb{R} \to \mathbb{R}$ be the zero function, and this time consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \chi_{[n, \infty)}(x).$$
Note that $f_n$ is a simple function and $[n, \infty)$ is measurable by Theorem 3.12. (p. 61, Royden) because $[n, \infty)$ is closed. By definition
\[ \int f_n = m([n, \infty)) = \infty \]
for every $n \in \mathbb{Z}^+$. 

Now we'll show that $f_n$ is monotone decreasing. Let $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. If $x < n + 1$, then $x \notin [n + 1, \infty)$ so
\[ f_{n+1}(x) = 0 \leq f_n(x). \]
If $x \geq n + 1$, then $x \in [n + 1, \infty)$ and $x \in [n, \infty)$, so
\[ f_{n+1}(x) = 1 \leq 1 = f_n(x). \]
This establishes that $f_{n+1} \leq f_n$ for all $n \in \mathbb{Z}^+$. In other words $\{f_n\}$ is a monotone decreasing sequence of functions.

Hence
\[ \int f = \int 0 = 0 \neq \infty = \lim_{n \to \infty} \infty = \lim_{n \to \infty} \int f_n \]
demonstrating that the Monotone Convergence Theorem need not hold for decreasing sequences.
4.8. Prove the following generalization of Fatou’s Lemma: if \( \{f_n\} \) is a sequence of non-negative functions, then

\[
\int \lim f_n = \lim \int f_n.
\]

**Proof.**

Define the sequence of functions \( \{g_n\} \) by the rule

\[
g_n(x) = \inf_{k \geq n} f_k(x)
\]

Let \( n \in \mathbb{Z}^+ \), \( x \in \mathbb{R} \). Clearly \( \{k \in \mathbb{Z}^+ : k \geq n + 1\} \subseteq \{k \in \mathbb{Z}^+ : k \geq n\} \). This means that

\[
g_n(x) = \inf_{k \geq n} f_k(x) \leq \inf_{k \geq n+1} f_k(x) = g_{n+1}(x).
\]

Note that

\[
\lim f_n(x) = \sup_n \inf_{k \geq n} f_k(x) = \lim_{n \to \infty} \inf_{k \geq n} f_k(x) = \lim_{n \to \infty} g_n(x)
\]

and that

\[
g_n(x) = \inf_{k \geq n} f_k(x) \leq f_n(x).
\]

As our choice of \( n \) and \( x \) were arbitrary, this establishes that \( \{g_n\} \) is a monotone increasing sequence (since \( g_n \leq g_{n+1} \)), that \( g_n \to \lim f_n \) and that \( g_n \leq f_n \) for every \( n \in \mathbb{Z}^+ \).

By Theorem 3.20 (p. 68, Royden) each \( g_n \) is measurable. By the Monotone Convergence Theorem and

\[
\int \lim f_n = \lim_{n \to \infty} \int g_n
\]

Because \( \{ \int g_n \} \) is a convergent sequence we have

\[
\lim_{n \to \infty} \int g_n = \lim \int g_n.
\]

It is easy to see that each \( g_n \) is nonnegative. By Proposition 4.8 (p. 85, Royden) \( \int g_n \leq \int f_n \) for all \( n \in \mathbb{Z}^+ \). Hence

\[
\int \lim f_n = \lim \int g_n \leq \lim \int f_n
\]

as desired.

\[\blacksquare\]
4.9. Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on \((-\infty, \infty)\) such that \( f_n \to f \) a.e., and suppose that \( \int f_n \to \int f < \infty \). Then for each measurable set \( E \) we have \( \int_E f_n \to \int_E f \).

**Proof.**

Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on \((-\infty, \infty)\) such that \( f_n \to f \) a.e., and suppose that \( \int f_n \to \int f < \infty \). Let \( E \) be a measurable set.

By Fatou’s Lemma (p. 86, Royden) \( \int_E f \leq \liminf \int_E f_n \). Now we’ll show that \( \int_E f \geq \lim \int_E f_n \). Note that \( E, E^c \) are measurable, so by Proposition 12 (p. 87, Royden)

\[
\int f = \int_E f + \int_{E^c} f \quad \text{and} \quad \int f_n = \int_E f_n + \int_{E^c} f_n
\]

By Proposition 4.12 (p. 87, Royden) and the fact that \( \{\int f_n\} \) is convergent

\[
\int f + \int_{E^c} f = \int f = \lim_{n \to \infty} \int f_n = \liminf \int f_n = \lim \left( \int_E f_n + \int_{E^c} f_n \right).
\]

Equivalently,

\[
\int_E f = \lim \left( \int_E f_n + \int_{E^c} f_n \right) - \int_{E^c} f.
\]

Now \( E^c \) is also measurable, so by Fatou’s Lemma

\[
\int_{E^c} f \leq \lim \int_{E^c} f_n
\]

or equivalently

\[
-\int_{E^c} f \geq -\lim \int_{E^c} f_n = \lim \left( -\int_{E^c} f_n \right).
\]

Hence

\[
\int f = \lim \left( \int_E f_n + \int_{E^c} f_n \right) - \int_{E^c} f
\]

\[
\geq \lim \left( \int_E f_n + \int_{E^c} f_n \right) - \lim \left( \int_{E^c} f_n \right)
\]

\[
\geq \lim \left( \int_E f_n + \int_{E^c} f_n - \int_{E^c} f_n \right)
\]

\[
= \lim \int_E f_n
\]

This establishes that

\[
\lim \int_E f_n \leq \int f \leq \lim \int_E f_n.
\]
Because \( \lim \int_E f_n \leq \overline{\lim} \int_E f_n \) always holds, it follows that

\[
\overline{\lim} \int_E f_n = \int_E f = \lim \int_E f_n
\]

or equivalently that

\[
\int_E f = \lim_{n \to \infty} \int_E f_n.
\]