1. Rotations

**Exercise 1.** Recall that a *isometry* is a map $R : \mathbb{C} \to \mathbb{C}$ which preserves distances, that is such that for all $z_1, z_2 \in \mathbb{C}, |z_1 - z_2| = |Rz_1 - Rz_2|$. Show that (a) the composition of two isometries is an isometry, (b) the inverse of an isometry is an isometry.

**Exercise 2.** Show that $R(z) = \rho z + t$ where $\rho = e^{i\alpha}, 0 \leq \alpha < 2\pi$ is an isometry ($t \in \mathbb{C}$).

**Exercise 3.** Show that the map $R : \mathbb{C} \to \mathbb{C}, R(z) = \rho \bar{z}, |ho| = 1$ is an isometry.

**Exercise 4.** Let $T$ be a piecewise rotation with two triangular atoms, $P_0$ and $P_1$. Let $T|_{P_0} = R_0$ and $T|_{P_1} = R_1$. Write down the set $H_{11}$ of all points whose itineraries start with 10 as the intersection of two sets (two triangles). What are the possible shapes and the number of vertices of $H_{11}$?

**Exercise 5.** A solution of $z^p = 1$ is called a $p$-th root of unity. Show that if $\zeta$ is a 10-th root of unity, then either $\zeta$ or $-\zeta$ is a 5-root of unity. What are other numbers $p$ with the property that if $\zeta$ is a $(2p)$-th root of unity, then either $\zeta$ or $-\zeta$ is a $p$-root of unity?

**Exercise 6.** Recall the definition of the return map: Let $f : X \to X$ and $U \subset X$. Let $\Delta$ be a subset of $U$ be such that for all $x \in \Delta$, there is $n$ such that $f^n(x) \in U$. Given an $x \in \Delta$, let $\tau(x)$ be the smallest $n$ such that $f^n(x) \in U$. The first return map of $f$ to $U$ is defined as $f_U = f^{\tau(x)}x$.

- (a) Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x$. Determine the return map $f_{[0,1]}$ and its domain. (b) Let $f : [0,1) \to [0,1), f(x) = x + 1/3 \mod 1$. Determine the return map $f_{[0,1/2]}$. Hint: The return map is a piecewise translation.

**Exercise 7.** Let $\sigma$ be the substitution

\[
\sigma : \begin{array}{c c c}
0 & \rightarrow & 01 \\
1 & \rightarrow & 1.
\end{array}
\]

What are the fixed points of $\sigma$ (The map $\sigma$ acts of the space of binary infinite sequences). Are there periodic orbits under $\sigma$ of period greater than one?