1. Integral Question.

In this note, I will explain how to integrate \( \int_{0}^{\pi} \frac{dx}{\cos^4 x + \sin^4 x} \) symbolically.

First, before we apply trigonometric substitutions, we will reduce the powers in the denominators using the following well-known formula. \( \cos 2y = 1 - 2\sin^2 y = 2\cos^2 y - 1 \). From the above formula, we obtain, \( \sin^2 y = \frac{1 - \cos 2y}{2} \) and \( \cos^2 y = \frac{1 + \cos 2y}{2} \).

We will apply formulas above twice, first for \( y = x \), then for \( y = 2x \).

\[
\int_{0}^{\pi} \frac{dx}{\cos^4 x + \sin^4 x} = \int_{0}^{\pi} \frac{dx}{(\cos^2 x)^2 + (\sin^2 x)^2} = \int_{0}^{\pi} \frac{dx}{(\cos 2x + 1)^2 + (\frac{1 - \cos 2x}{2})^2} = \\
= \int_{0}^{\pi} \frac{4dx}{2\cos^2 2x + 2} = \int_{0}^{\pi} \frac{2dx}{\cos 2x + 1} = \int_{0}^{\pi} \frac{2dx}{\cos 2(2x) + 1} + 1 = \int_{0}^{\pi} \frac{4dx}{\cos 4x + 3} = \int_{0}^{4\pi} \frac{dz}{\cos z + 3}.
\]

In the last integral, we used the substitution, \( z = 4x \). Now we will use the trig substitution. In order for us to use the trigonometric substitution, \( t = \tan(\frac{z}{2}) \) \( (dz = \frac{2}{t^2 + 1} dt) \), we need to restrict the domain of \( z \) only to intervals on which \( \tan(\frac{z}{2}) \) is a continuous function. Hence, we will need to split the interval \((0, 4\pi)\) into four intervals: \((0, \pi), (\pi, 2\pi), (2\pi, 3\pi), (3\pi, 4\pi)\).

We compute the integral over the first interval. We obtain:

\[
\int_{0}^{\pi} \frac{dz}{\cos z + 3} = \int_{0}^{\pi} \frac{dz}{\cos z + 3} = \int_{0}^{\infty} \frac{2t^2}{t^2 + 1} dt = \int_{0}^{\infty} \frac{2dt}{4t^2 + 2} = \int_{0}^{\infty} \frac{dt}{2t^2 + 1} = \int_{0}^{\infty} \frac{\sqrt{2}dt}{v^2 + 1} = \\
= \lim_{p \to \infty} \frac{\sqrt{2}}{2} [\arctan(v)]^p_0 = \frac{\sqrt{2}\pi}{4}.
\]

The integral over three remaining intervals is identical. Hence our final answer is 4 times the answer above, that is \( \sqrt{2}\pi \). This is the answer most of you obtained using numerical integration.

Note that this example, if not considered carefully, may be tricky. If you try to evaluate the integral \( \int \frac{dx}{\cos^4 x + \sin^4 x} \) symbolically in Mathematica, you will get the following antiderivative:

\[
\int \frac{dx}{\cos^4 x + \sin^4 x} = -\arctan \left[ \sec[x] \left( \cos[x] - \sqrt{2}\sin[x] \right) \right] + \arctan \left[ \sec[x] \left( \cos[x] + \sqrt{2}\sin[x] \right) \right].
\]

This is correct on any interval on which the expression on the right is continuous. However, this expression is not continuous on the interval \((0, \pi)\) (plot it). Hence, you cannot apply the Fundamental Theorem of Calculus to the above expression in order to evaluate \( \int_{0}^{\pi} \frac{dx}{\cos^4 x + \sin^4 x} \). For this reason, as you noticed, the command

\text{Integrate}[1/(\cos[x]^4 + \sin[x]^4), \{x, 0, \pi\}]
\]

gives an incorrect answer.

One lesson to be learned from this example is that, when we deal with antiderivatives whose expressions are discontinuous, we need to exercise caution.