I am a combinatorialist with very broad mathematical interests. Much of my research is devoted to bridging the artificial gap between “pure” questions in algebra, geometry, and topology, and “applied” questions in combinatorics, phylogenetics, and optimization, by studying the rich discrete structures that they have in common.

Much of my recent work involves four of my strongest recent interests: the geometry of flags, Bergman complexes, satisfiability problems, and Tutte-Grothendieck invariants. I describe my main results and future directions in these areas in Sections 1, 2, 3, and 4, respectively. Additionally, I am always excited to learn about and become involved in new research projects.

1 Geometry of flags.

In ongoing work with Sara Billey [13], we are exploring the connections between the geometry of $d$ flags in $\mathbb{C}^n$, certain tiling problems in $\mathbb{R}^{d-1}$, and a tropical analog of oriented matroids.

Let $E_\bullet = \{\{0\} = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n\}$ denote a complete flag in $\mathbb{C}^n$, where $E_i$ is a vector space of dimension $i$. Consider $d$ generic flags $E^1_\bullet, \ldots, E^d_\bullet$, and the arrangement of lines of the form $E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d}$, obtained by intersecting them. Let $T_{n,d}$ be the matroid that keeps track of the linear dependence relations among these lines.

Let $T_{n,3}$ be a triangular array of dots with $n$ dots on each edge. Let $T_{n,d}$ be the analogous $(d-1)$-dimensional array. There is a one-to-one correspondence between our lines and the dots in $T_{n,d}$, which helps us describe the matroid $T_{n,d}$ combinatorially:

**Theorem 1.1.** [13] A set $S$ of dots in $T_{n,d}$ is a basis for the matroid $T_{n,d}$ if and only if no copy of $T_{k,d}$ inside $T_{n,d}$ contains more than $k$ dots of $S$.

1.1 Tilings and triangulations.

Consider the lozenge tilings of an upward facing equilateral triangle $T(n)$ of size $n$ into unit rhombi and upward unit triangles. There are exactly $n$ triangles in such a tiling.

**Theorem 1.2.** [13] The possible locations of the $n$ triangles in a lozenge tiling of $T(n)$ are precisely the bases of $T_{n,3}$.

From Theorem 1.2, after some graph theoretic observations (Menger’s theorem and Lindström’s lemma), we obtain Theorem 1.3, will be important to us later. These ideas also led me to a new proof of the duality between transversal and cotransversal matroids [12], which sheds light on the duality between the theorems of Menger and König in graph theory.

**Theorem 1.3.** [13] There is a simple representation of $T_{n,3}$ as a collection of vectors described by certain weighted paths in the triangular grid.

The $(d-1)$-dimensional analogs of lozenge tilings are the fine mixed subdivisions of the simplex $n\Delta_{d-1}$ of size $n$. Now the tiles are $(d-1)$-dimensional products of simplices; in three dimensions, they are unit tetrahedra, triangular prisms, and parallelepipeds. We pay special attention to the tiles which are simplices.

**Theorem 1.4.** [13] In any fine mixed subdivision of $n\Delta_{d-1}$ there are exactly $n$ simplices. The locations of those $n$ simplices form a basis of $T_{n,d}$.
Conjecture 1.5. [13] Every basis of $T_{n,d}$ is the set of locations of the simplices in some fine mixed subdivision of $n\Delta_{d-1}$.

Given the correspondence between coherent fine mixed subdivisions of $n\Delta_{d-1}$, regular triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$, and combinatorial types of arrangements of $d$ generic tropical hyperplanes in tropical $(n-1)$-space [25, 37], Conjecture 1.5 is an invitation to study more closely those combinatorial types. We do this in an ongoing project with Sara Billey and Mike Develin; we develop the theory of tropical oriented matroids, which axiomatize the combinatorial types of the arrangements of tropical pseudohyperplanes, as well as the subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$. Such a theory should allow us to prove Conjecture 1.5, as well as the following:

Conjecture 1.6. The triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are connected by flips.

Diaconis and Sturmfels [26, 43] showed that the polytope $\Delta_{n-1} \times \Delta_{d-1}$ is intimately related to sampling and optimization for contingency tables and transportation problems; we plan to explore the repercussions of tropical oriented matroids in these problems.

1.2 Schubert calculus.

The Schubert calculus was the original motivation for this project. Recall that the relative position of two flags $E_\bullet$ and $F_\bullet$ in $\mathbb{C}^n$ is described by a permutation $w \in S_n$. Billey and Vakil [17] discovered a method for explicitly computing all flags which are in relative positions $u, v,$ and $w$ with respect to three given Schubert-generic flags $E_\bullet, F_\bullet,$ and $G_\bullet$. The number of them, $c_{uvw}$, does not depend on the flags: it is a structure constant in the cohomology ring of the flag manifold.

Recently, much work has been devoted to gaining a solid understanding of these numbers, similar to our understanding of the analogous Littlewood-Richardson coefficients for the Grassmannian.

The input for Billey and Vakil’s computation consists of the permutations $u, v,$ and $w$, and a vector on each line $E_i \cap F_j \cap G_k$. These vectors are a representation of the matroid $T_{n,3}$ of Theorem 1.1. In practice, to compute $c_{uvw}$ using this method, we need this representation to be simple enough to allow computer manipulation, but complicated enough for the corresponding flags $E_\bullet, F_\bullet,$ and $G_\bullet$ to be Schubert-generic. We claim that we have achieved this in Theorem 1.3.

Conjecture 1.7. [13] The representation of $T_{n,3}$ given in Theorem 1.3 is Schubert-generic.

If Conjecture 1.7 is true, it will give us a way of computing the structure constants $c_{uvw}$ in terms of paths in a triangular grid, without reference to an initial set of flags. This is not only helpful computationally; it is also a promising step in the direction of solving one of the long-standing open problems in the Schubert calculus:

Problem 1.8. Find a combinatorial description for the structure constants $c_{uvw}$ of the cohomology ring of the flag manifold.

Additionally, a consequence of our work is a very simple criterion for guaranteeing that certain $c_{uvw}$’s vanish. We are working on comparing this with similar results due to Knutson, Lascoux and Schutzenberger, and Purbhoo. We have verified that our criterion, with very minor variations, is enough to detect all vanishing structure constants for $n \leq 5$. We are also developing a systematic way of fine-tuning this method for higher $n$. The following question in geometric transversal theory lies at the heart of this matter:

Question 1.9. Suppose we are given a collection of subspaces of $\mathbb{C}^n$, and we know the dimensions of their various intersections. If we choose a non-zero vector in each subspace, how small can we make the rank of the chosen vectors?
2 Bergman complexes, phylogenetic trees, and graph associahedra.

Two related mathematical species, known as amoebas and Bergman complexes, have recently arisen independently in several different areas, such as real algebraic geometry, complex analysis, and dynamical systems. I am very interested in understanding the second of these species combinatorially and topologically.

The amoeba of an algebraic variety $X \in \mathbb{C}^n$ is the following subset of $\mathbb{R}^n$:

$$\mathcal{A}(X) = \text{Log} X = \{(\log |z_1|, \ldots, \log |z_n|) : z \in X \cap (\mathbb{C}^*)^n\}.$$

It is very difficult to say what amoebas looks like. We focus our attention on the Bergman complex $B(X)$: a subset of the unit sphere which, roughly speaking, consists of the directions in which $\mathcal{A}(X)$ goes to infinity. This object, also known as a tropical variety, logarithmic limit set, or Bieri-Groves set, is in itself important from several points of view. Fortunately, it is also much simpler: it is a pure $(\dim X - 1)$-dimensional spherical polyhedral complex [14, 15].

The results in Section 2 are motivated by the following question of Sturmfels:

**Question.** [43] What interesting things can be said about the combinatorial and topological structure of the Bergman complex of a linear variety?

I wish to convey that, even though we are restricting our attention to linear varieties, the answer to this question is:

**Answer.** [5, 6, 7, 9] Much more than one might expect!

Bergman complexes are an extremely rich source of interesting mathematics. Combinatorial and matroid theoretic tools are at the core of our understanding of their topology. In turn, these complexes have motivated very interesting questions and answers, in connection with objects such as the space of phylogenetic trees, the Whitehouse complex, De Concini and Procesi’s compactification of the complement of a subspace arrangement, Coxeter complexes, and graph associahedra.

Sections 2.1 and 2.3 are joint work with Carly Klivans, Section 2.2 is joint with Carly Klivans and Lauren Williams, and Section 2.5 is joint with Vic Reiner and Lauren Williams. Our results have proved useful in Feichtner and Sturmfels’ work on nested set complexes [28], their work with Dickenstein on tropical discriminants [27], and Speyer’s work on tropical linear varieties [39], among others.

2.1 The Bergman complex of a matroid.

The Bergman complex $\mathcal{B}(V)$ of a subspace $V$ depends only on the matroid $M_V$ associated to $V$ under a fixed coordinate system. The language of matroid theory is most convenient for describing our results.

Given a loopless matroid $M$ on the ground set $E$ and a weight vector $\omega \in \mathbb{R}^E$, let $M_\omega$ be the matroid whose bases are the bases of $M$ of minimum $\omega$-weight. The Bergman complex of $M$ is:

$$\mathcal{B}(M) = \{\omega \in \mathbb{R}^E : M_\omega \text{ is loopless}\} \cap S^{E-1}.$$

If $V$ is a linear subspace of $\mathbb{C}^n$ and $M_V$ is the associated matroid, then the geometric definition of $\mathcal{B}(V)$ in the previous section and the combinatorial definition of $\mathcal{B}(M_V)$ coincide.
Recall that the flats of a matroid $M$, partially ordered by containment, form a lattice $L_M$. Let $\mathcal{L}_M = L_M - \{\emptyset, 1\}$. The order complex of $\mathcal{L}_M$, denoted $\Delta(\mathcal{L}_M)$, is the simplicial complex whose vertices are the elements of $\mathcal{L}_M$, and whose faces correspond to the chains in $\mathcal{L}_M$. This simplicial complex is very well understood topologically: it is pure, shellable, and homotopy equivalent to a wedge of $\mu(L_M) \cdot (r(M) - 2)$-dimensional spheres.

Somewhat surprisingly, the Bergman complex $\mathcal{B}(M)$ is very closely related to $\Delta(\mathcal{L}_M)$, and this relationship allows us to describe its topology.

**Theorem 2.1.** [5] $\Delta(\mathcal{L}_M)$ is a subdivision of $\mathcal{B}(M)$. Consequently, $\mathcal{B}(M)$ is homotopy equivalent to a wedge of $\mu(L_M) \cdot (r(M) - 2)$-dimensional spheres.

The nice topology of Bergman complexes suggests the following question, an answer to which would reveal more about their combinatorial structure.

**Question 2.2.** Is the Bergman complex $\mathcal{B}(M)$ shellable?

### 2.2 The positive Bergman complex of an oriented matroid.

In [41], Speyer and Williams defined the positive part of a tropical variety, in analogy with the familiar notion of total positivity. The Bergman complex of a linear variety can be regarded as a tropical variety, and [7] is devoted to studying its positive part.

If $M$ is an acyclic oriented matroid, we define the *positive Bergman complex* of $M$ to be:

$$\mathcal{B}^+(M) = \{ \omega \in \mathbb{R}^E : M_{\omega} \text{ is acyclic} \} \cap S^{E-1} \cap S_{E-1}.$$ 

If $V$ is a linear subspace of $\mathbb{C}^n$ and $M_V$ is the associated oriented matroid, then the geometric definition of the positive part of $\mathcal{B}(V)$ coincides with the combinatorial definition of $\mathcal{B}^+(M_V)$. Recall that the positive flats of a matroid $M$, partially ordered by containment, form a lattice $\mathcal{F}_{fv}(M)$: the Las Vergnas face lattice of $M$.

**Theorem 2.3.** [7] $\Delta(\mathcal{F}_{fv}(M))$ is a subdivision of $\mathcal{B}^+(M)$. Consequently, $\mathcal{B}^+(M)$ is homotopy equivalent to an $(r(M) - 2)$-dimensional sphere.

### 2.3 The complex of phylogenetic trees is a Bergman complex.

The complex of trees or Whitehouse complex $T_n$ is a ubiquitous simplicial complex, which was defined by Boardman in the language of homotopy theory [19], and later studied in connection with several different objects, such as the space of phylogenetic trees [16], the moduli space of nonsingular curves of genus 0 [29], and the WDVV equations of string theory [36]. Surprisingly, the Bergman complex provides yet a different point of view on this object, and allows us to better understand its topology.

Let $M(K_n)$ be the graphical matroid of the complete graph $K_n$: its lattice of flats is known to be the lattice $\Pi_n$ of partitions of $n$, ordered by refinement.

**Theorem 2.4.** [5] The complex of trees $T_n$ is equal to the Bergman complex of $M(K_n)$. Consequently, $\Delta(\Pi_n)$ is a subdivision of $T_n$.

Previously, several authors had given different proofs of the fact that $T_n$ is homotopy equivalent to a wedge of $(n-1)! \cdot (n-3)$-dimensional spheres; $\Delta(\Pi_n)$ was known to have the same homotopy type. Furthermore, the representations of $S_n$ on the homology groups of these two complexes were known to be isomorphic. Theorem 2.4 provides a conceptual explanation of these results.

We also have a nice description of the positive Bergman complex in this case.
Theorem 2.5. [7] For any acyclic orientation of $K_n$, the positive part of the Bergman complex of $M(K_n)$ is dual to the associahedron $A_{n-2}$.

2.4 The metric geometry of the space of phylogenetic trees.

As mentioned in Section 2.3, the fan over the Bergman complex $B(K_n)$ can be regarded as a space of phylogenetic trees; i.e., rooted trees with lengths associated to the edges. In one application, the leaves of the tree represent the present day species in the tree of evolution. Consider the following very natural question. Suppose that we do not know the evolutionary tree of certain taxa, but we are able to experimentally (approximately) predict the pairwise distances between them, say, by comparing their DNA sequences. How do we find the phylogenetic tree that most closely matches those measured distances? If “closeness” is measured in the $\ell_p$ metric for any $p > 0$, this question is known to be NP-hard. For the $\ell_\infty$ metric, Chepoi and Fichet [22] found a method for constructing the closest tree.

In [6], I show that their construction has a natural explanation in the context of tropical geometry and matroid theory. I prove that the space of phylogenetic trees $T_n$ is a tropical polytope. The point in $T_n$ closest to the vector $v$ of measured pairwise distances is the tropical projection of $v$ onto $T_n$. This is the geometric content of Chepoi and Fichet’s construction.

By combining this tropical geometric observation with Tarjan’s generalization of the greedy algorithm for optimization in a matroid [31], I obtained a more general and efficient algorithm for constructing the phylogenetic tree which best fits measured data.

This work was done while I was involved with the mathematical biology research groups at the University of California, Berkeley and San Francisco State University in 2004, and I am looking forward to collaborating with them again this spring.

2.5 Coxeter arrangements, wonderful models, and graph associahedra.

Let $\Phi$ be a (possibly infinite, possibly non-crystallographic) root system, and let $M_\Phi$ be its vector matroid. To describe the Bergman complex of $M_\Phi$ and its positive part, we recall two constructions associated to $\Phi$.

Given a complex subspace arrangement $\mathcal{A}$, De Concini and Procesi [24] constructed a wonderful arrangement model, useful for studying the topology of the complement of $\mathcal{A}$. It is obtained by blowing up the non-normal crossings of the arrangement, without changing its complement. The nested set simplicial complex of $\mathcal{A}$ encodes the underlying combinatorics of their construction.

We are interested in this construction when applied to the Coxeter arrangement $A_\Phi$, consisting of the hyperplanes normal to the roots of $\Phi$.

Theorem 2.6. [9] If $\Phi$ is finite, the Bergman complex of $M_\Phi$ is the nested set complex of $A_\Phi$.

Now let $D$ be the Dynkin diagram of $\Phi$. Define a tube to be a connected subgraph of $D$, and say a set of tubes is a tubing if any two tubes are either nested, or disjoint and non-adjacent. These tubings form a poset by containment. Carr and Devadoss [21], Davis, Januszkiewicz, and Scott [23], and Postnikov [33] independently discovered the graph associahedron of type $\Phi$: a polytope whose face poset is isomorphic to the poset of tubings of $D$.

Theorem 2.7. [9] For any $\Phi$, the positive Bergman complex of $M_\Phi$ is dual to the graph associahedron of type $\Phi$.

This work motivated two further projects concerning the graph associahedron $A_G$ of a graph $G$. Firstly, I constructed a simple geometric realization of $A_G$. [11] Secondly, Loday constructed
a triangulation of the associahedron whose simplices are indexed by parking functions; Reiner, Williams, and I hope to generalize this construction to a triangulation of $A_G$, whose simplices are indexed by Postnikov and Shapiro’s $G$-parking functions. Finally, we are working on settling the following conjecture, regarding the face numbers of the graph associahedron. Recall that the $h$-polynomial of a polytope $P$ is a way of encoding the number of faces of each dimension of $P$.

**Conjecture 2.8.** For each simple graph $G$ on vertex set $[n]$, there exists a subset $S_n(G)$ of the permutations in $S_n$ such that the $h$-polynomial of the graph associahedron $A_G$ is the generating function for the permutations in $S_n(G)$ which keeps track of the number of descents.

Conjecture 2.8 fits nicely within the vast literature on face enumeration in polytopes and other cell complexes. We have verified it for several non-trivial families of graphs.

From the geometry of Bergman complexes, we obtain a map $f_G$ from $S_n$ to the maximal tubings of $G$. We further conjecture that $S_n(G)$ can be chosen to contain exactly one preimage of each maximal tubing. This suggests a close connection with Björner and Wachs’ work on the map from the weak Bruhat order $S_n$ to the Tamari lattice $T_n$ [18], which is also our map $f_{K_n}$.

## 3 Random $k$-SAT problems and convexity.

One of the most effective algorithms for solving random $k$-SAT problems in practice, even with densities close to threshold, is the survey propagation algorithm of Braunstein, Mezard, and Weigt [20], which has its roots in theoretical physics. In their effort to explain the efficiency of this algorithm, Maneva, Mossel, and Wainwright [32] encountered a ranked lattice associated to a SAT formula and a valid generalized assignment. They showed that the performance of the algorithm could be partially explained by the fact that these SAT lattices satisfy the propagation equality:

$$\sum_{x \in L} (1 - t)^{\# \text{ elements covered by } x} (r(L) - r(x)) = 1.$$  

They also conjectured that SAT lattices are distributive.

Studying these SAT lattices is the goal of a project with Elitza Maneva. [8] We proved that SAT lattices satisfy a stronger multivariate propagation equality, which in fact characterizes them.

**Theorem 3.1.** [8] A ranked lattice is SAT if and only if it satisfies the multivariate propagation equality.

We also showed that distributive lattices do satisfy the multivariate propagation equality. However, the following characterization disproves Maneva, Mossel, and Wainwright’s conjecture.

**Theorem 3.2.** [8] The SAT lattices are precisely the join-distributive lattices. The 2-SAT lattices are precisely the distributive lattices.

Join-distributive lattices are well understood objects; they are equivalent to convex geometries [30], certain objects which abstract the combinatorial properties of convexity. In the near future, we hope to understand the consequences of this theory of convexity on the satisfiability of random $k$-SAT problems near the threshold density.
4 Tutte-Grothendieck invariants.

The Tutte polynomial of a matroid \( M \) with ground set \( S \) and rank function \( r \) is given by

\[
T_M(x, y) = \sum_{A \subseteq S} (x - 1)^{r(A)} (y - 1)^{|A| - r(A)}.
\]

Much of the interest in the Tutte polynomial derives from the fact that it is the universal generalized Tutte-Grothendieck or \( T-G \) invariant. This means that any invariant \( f(M) \) which satisfies a certain type of recursion is an evaluation of it.

The Tutte polynomial is of great power and applicability, because many interesting quantities in areas such as combinatorics, graph theory, statistical mechanics, and knot theory, are generalized T-G invariants. Some examples are the chromatic and flow polynomials of a graph, the Jones polynomial of an alternating knot, and the partition function of the \( q \)-state Potts model.

There is a great imbalance between the wide variety of applications and the relatively small existing theory of the Tutte polynomial. The characteristic polynomial of a matroid is well understood from combinatorial, algebraic, topological and probabilistic points of view. One of my main research projects has been to develop a similar background for the Tutte polynomial.

4.1 Hyperplane arrangements and their Tutte polynomials.

In \([3, 4]\), I show how to extend the definition of the Tutte polynomial to (possibly affine) hyperplane arrangements, and introduce a new finite field method, which computes Tutte polynomials by solving enumerative problems in the finite vector space \( \mathbb{F}_q^n \).

A \( \mathbb{Z} \)-arrangement in \( \mathbb{R}^n \) is an arrangement whose defining equations have integer coefficients. Such an arrangement \( \mathcal{A} \) induces an arrangement \( \mathcal{A}_q \) over the finite vector space \( \mathbb{F}_q^n \) for any prime power \( q \), in the obvious way. It is convenient to state the result in terms of the coboundary polynomial \( \chi_{\mathcal{A}}(q, t) = (t - 1)^r T_{\mathcal{A}}(\frac{2t + 1}{t - 1}, t) \).

**Theorem 4.1.** \([3]\) Let \( \mathcal{A} \) be a \( \mathbb{Z} \)-arrangement of rank \( r \) in \( \mathbb{R}^n \). If \( q \) is a power of a large enough prime, and \( h(p) \) denotes the number of hyperplanes of \( \mathcal{A}_q \) that \( p \) lies on, then

\[
q^{n-r} \chi_{\mathcal{A}}(q, t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}.
\]

I use this technique to compute the Tutte polynomial of several arrangements of interest, such as Coxeter arrangements. Various classical combinatorial objects, such as parking functions, Dyck paths, and alternating trees, are known to be related to certain deformations of the braid arrangement. These ideas are used to give unusual new formulas for the generating functions of these objects. An example of the type of result obtained is the following.

**Theorem 4.2.** \([3]\) Let \( a_n \) be the number of alternating trees on \([n + 1]\); i.e., labelled trees where every vertex is either larger or smaller than all of its neighbors. Let

\[
\frac{e^{x(1+y)} + y}{e^{x(1+y)} - y^2} = \sum_{r \geq 0} A_r(x)y^r.
\]

Then, for each positive integer \( n \),

\[
\frac{A_{n-1}(x)}{A_n(x)} = a_0 + a_1 x + \cdots + a_n \frac{x^n}{n!} + O(x^{n+1}).
\]
4.2 The h-vector and an ideal generated by powers of linear forms.

In joint work with Alex Postnikov [10], we investigate an algebra associated to a collection of vectors in a vector space. This is a variant of two algebras with beautiful enumerative properties, previously studied by Postnikov, Shapiro, and Shapiro. [34, 35]

Consider a set of vectors $E = \{v_1, \ldots, v_N\}$ in an $n$-dimensional vector space $V$ over a field $k$, and let $M_E$ denote the matroid associated to it. Let $T_{M_E}(1, y) = h_{N-n} + h_{N-n-1}y + \cdots + h_0y^{N-n}$; the $h$s are commonly known as the $h$-vector of $M_E$.

Now let $Sym(V^*)$ be the symmetric algebra of the dual vector space $V^*$. A hyperplane spanned by some of the vectors in $E$ is simply said to be spanned by $E$. For each such hyperplane $H$, let $\lambda_H$ be the corresponding element of $V^*$, and let $d(H)$ be the number of vectors of $E$ which do not lie on $H$. Define the graded algebra

$$\mathcal{H}_E = Sym(V^*) / \langle \lambda_H^{d(H)} : H \text{ is a hyperplane spanned by } E \rangle.$$ 

**Theorem 4.3.** [10] The Hilbert polynomial of the algebra $\mathcal{H}_E$ depends only on the matroid $M_E$; it is equal to $y^{N-n}T_{M_E}(1, 1/y) = h_0 + h_1y + \cdots + h_{N-n}y^{N-n}$.

**Conjecture 4.4.** For each $k$, the Hilbert polynomial of $\mathcal{H}_{E,k} = Sym(V^*) / \langle \lambda_H^{d(H)+k} \rangle$ is a generalized Tutte-Grothendieck invariant for the matroid of $E$.

We already know that $\text{Hilb}(\mathcal{H}_{E,0}, t)$ and $\text{Hilb}(\mathcal{H}_{E,1}, t)$ are T-G invariants [10, 35]. An affirmative answer to Conjecture 4.4 is likely to give us a new description of the Tutte polynomial.

The polynomial $T_M(1, y)$ arising in Section 4.2 is of great interest in itself. Its coefficients form the $h$-vector of $M^*$, which is the subject of one of the most intriguing open problems in matroid theory:

**Conjecture 4.5.** (Stanley, 1977) [42] If $(h_0, \ldots, h_d)$ is the $h$-vector of a matroid, then there exists a pure multicomplex $\Gamma$ such that $h_i$ is the number of monomials in $\Gamma$ of degree $i$.

This conjecture has inspired a great amount of work and increasingly stronger results towards the ultimate goal of describing the possible $h$-vectors of a matroid. Theorem 4.3 gives a very exciting new perspective on it, which I will explore in the near future.

4.3 The Catalan matroid.

Consider all Dyck paths of length $2n$: paths in the plane from $(0, 0)$ to $(2n, 0)$, with steps $(1, 1)$ and $(1, -1)$, that never go below the x-axis. Each Dyck path $P$ defines an up-step set, consisting of the integers $i$ for which the $i$-th step of $P$ is $(1, 1)$.

**Theorem 4.6.** [2] The up-step sets of all Dyck paths of length $2n$ form the collection of bases of a matroid, the Catalan matroid $C_n$.

**Theorem 4.7.** [2] For a Dyck path $P$, let $a(P)$ denote the number of up-steps that $P$ takes before its first down-step, and let $b(P)$ denote the number of times that $P$ bounces on the x-axis. Then the Tutte polynomial of the Catalan matroid is equal to

$$T_{C_n}(x, y) = \sum_P x^{a(P)}y^{b(P)},$$

summing over all Dyck paths of length $2n$.

A nice enumerative corollary of Theorem 4.7 is that the joint distribution of $a(P)$ and $b(P)$ on Dyck paths is symmetric, because the Catalan matroid is self-dual.
References


