Proof (a) \( L(P) \) has a meet: \( F \land G = F \cap G \).

Any point with a meet and a \( \land \) has a join:

\[ F \lor G = \bigwedge (\text{upper bounds of } F \text{ and } G) \]

(Non-empty)

Graded: soon.

(b) \([0, G] \) is \( L(G) \)
\([F, G] \) is \( L(G \setminus F) \)

\( [v, G] \) is \( L(G \setminus v) \)

\( [v, G] \) is \( L(G \setminus v) / vs \)

Graded: soon.

\( \exists [F, G] \) is \( L(G \setminus F) / s \) \( \cap \) G/F

Note: \( \dim G/F = \dim G - \dim F - 1 \)

(a) Consider a maximal chain \( F \subset F_1 \subset \ldots \subset F_k \subset G \).

Suppose I skipped a dimension between \( F_i, F_{i+1} \).

Then \( \exists \text{ a maximal chain between } \exists F_i, F_{i+1} \).

So \( F_i / F_{i+1} \) is a vertex. Add it to chain!

(c) Follows from (b) since

\( [F, G] = L(1 - \text{polytope}) = L(\rightarrow) = \Diamond \)

(d) Follows from "polarity", our next goal.

Remark

\( P \) and \( Q \) are combinatorially isomorphic: \( (P \cong Q) \)

if their face lattices are isomorphic.

Note: This theorem puts very rigid requirements on \( L(P) \). These graphs have a lot of structure!

(Make on hw)

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**Polar Polytopes** (also called "dual polytopes")

\( P \subset \mathbb{R}^d \) polytope, or any set

The polar of \( P \) is

\[ P^\circ = \{ c \in (\mathbb{R}^d)^* : c \cdot x \leq 1 \text{ for all } x \in P \} \subset (\mathbb{R}^d)^* \]

**Theorem:** \( P, Q \) polytope,

(a) \( P \subseteq Q \implies P^\circ \subseteq Q^\circ \)

(b) \( P \subseteq P^\circ \)

(c) If \( 0 \in P \) then \( P = P^\circ \)

(d) If \( P = \text{conv } V \) then

\[ P^\circ = \{ q : q \cdot v \leq 1 \text{ for all } v \in V \} \]

(e) If \( P = P(A, 1) \),

\[ P^\circ = \{ cA : c \geq 0, c \cdot A = 1 \} = \text{conv } (\text{conv } A) \]

Sketch: (a), (b) are easy.

(c) \( \exists c \in P^\circ \) but \( \not\exists p \in P \)

\[ \text{let } c \cdot x = c \cdot y \leq 1 \]

\[ \text{let } c \cdot x = c \cdot y \leq 1 \]

(d) \( P^\circ = \{ q : q \cdot p \leq 1 \forall p \in P \} = \{ q : q \cdot v \leq 1 \forall v \in V \} \)

\( \subseteq \) trivial

\( \supseteq \) trivial

\( \exists \) achieves its max at a vertex,