The Fan Lattice

Recall: A poset \((P, \leq)\) is a set \(P\) with a partial order such that:
- \(x \leq x\)
- \(x \leq y, y \leq z \Rightarrow x \leq z\)
- \(x \leq y, y \leq x \Rightarrow x = y\)

Example: \((\mathbb{N}, \leq)\)

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0  1  2  3  4  5  6  7  8  9
\( \uparrow \)  \( \downarrow \)
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Boolean poset \(B_n = (2^n, \leq)\)

```
  0
/   \
1   2
  \mid
  \downarrow
  3
```

Fan poset \(L(P) = \{\text{faces of } P, \leq\}\)

- \(L(\{\}) = \{\}\)
- \(L(\Delta_{d-1}) = B_d\) (Exercise)

Chain: \(P_1 < P_2 < \ldots < P_n\) in \(P\) (length = \(k-1\))

- \(P\) is graded if all max chains from \(a\) to \(b\) have the same length
- If graded, it has “levels” or “ranks”
- \(P\) is a lattice if any \(x, y \in P\) have a least upper bound (meet) \(x \wedge y\) and a greatest lower bound (join) \(x \vee y\).
- (lattice: \(B_n\), not lattice: \(\Diamond\))

Theorem: \(P\) polytope
(a) \(L(P)\) is a lattice graded by \(\text{rk}(F) = \dim F + 1\)
(b) Every interval \([F, G]\) is also a face lattice.
(c) Every interval of length 2 is a diamond: \(\Diamond\)
(d) The opposite poset \(L(P)^{op}\) is also a face lattice.