Ex: $G_n: n$ lines in general position in $\mathbb{R}^2$

- No 2 parallel
- No 3 concurrent

$G_n \\ \downarrow \\ G_n$ 

$r(G_n) = r(G_{n-1}) + n$

$b(G_n) = b(G_{n-1}) + n - 1$

$\Rightarrow r(G_n) = \left(\begin{array}{c} n \\ 2 \end{array} \right) + n + 1$

$b(G_n) = \left(\begin{array}{c} n \end{array} \right) - n + 1$

Ex: $H_n: n$ planes in general position in $\mathbb{R}^3$

- No 2 parallel
- No 3 on a plane
- No 4 con

$H_n \\ \downarrow \\ H_n$ 

$r(H_n) = r(H_{n-1}) + \#(\text{old regions with } n - 1)$

$r(H_n) = r(H_n) + \left(\begin{array}{c} n \end{array} \right) + n + 1$

$b(H_n) = b(H_{n-1}) + \left(\begin{array}{c} n \end{array} \right) - n + 1$

$\Rightarrow r(G_n) = \left(\begin{array}{c} n \\ 2 \end{array} \right) + \left(\begin{array}{c} n \end{array} \right) + n + 1$

$b(G_n) = \left(\begin{array}{c} n \end{array} \right) - \left(\begin{array}{c} n \end{array} \right) + n - 1$

Prop: If $A^n = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^d$ is in general position, then

$r(A^n) = \left(\begin{array}{c} n \\ 2 \end{array} \right) + \sum_{i=1}^{n-1} \left(\begin{array}{c} n - i \\ d - 2 \end{array} \right)$

$b(A^n) = \left(\begin{array}{c} n \\ d \end{array} \right) - \sum_{i=2}^{n} \left(\begin{array}{c} d - 2 \\ i \end{array} \right)$

Proof: Indicate on $n+d$.

$r(A^n) = r(A_{n-1} \cup H_n) + r(A_{n-1} / H_n)$

$= r(A_{n-1}) + r(A_{n-1} / H_n)$

$= \left(\begin{array}{c} n-1 \\ 2 \end{array} \right) + \sum_{i=1}^{n-2} \left(\begin{array}{c} n-2 \\ d-2 \end{array} \right) + \left(\begin{array}{c} n-1 \\ d-2 \end{array} \right)$

$= \left(\begin{array}{c} n \\ 2 \end{array} \right) + \left(\begin{array}{c} n-1 \\ d-1 \end{array} \right) + \left(\begin{array}{c} n-1 \\ d-2 \end{array} \right)$

In particular: (check!)

$H_n: x_i = 0$ is even $\Rightarrow r(H_n) = \left(\begin{array}{c} n \\ d \end{array} \right) + \left(\begin{array}{c} n \\ d-2 \end{array} \right) = 2^{\phi(n,d)}$