In general:
\[ Z = V_1^+ + \cdots + V_n^+ \subset \mathbb{R}^d \]

Let \( c \in (\mathbb{R}^d)^* \). What is \( c \)-max face \( Z_c \)?

\[ Z_c = (V_1)_c + \cdots + (V_n)_c \]

\[ = \begin{cases} V_1^+ & \text{if } c \cdot V_1 > 0 \\ V_1^- & \text{if } c \cdot V_1 = 0 \\ V_1 & \text{if } c \cdot V_1 < 0 \end{cases} \]

So \( (c, c') \in N(Z) \) is the same as \( (c \cdot V_1, c \cdot V_1) \) having the same sign.

So we have a bijection:

\[
\begin{pmatrix}
\text{faces of } N(Z) \\
\text{faces of } Z(V)
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\text{nonempty sets of } V \\
\text{signed vectors of } V
\end{pmatrix}
\]

So we would do well understanding arrangements and their faces.

Note: not all sign vectors are realized as faces, e.g., \( +-, +-- \)

**Hyperplane Arrangement**

A hyperplane arrangement is \( \mathcal{A} = \{H_i\} \) in \( \mathbb{R}^d \) with

\[ H_i = \{x : \alpha_i \cdot x = b_i\} \]

If all \( b_i = 0 \), call it central.

"Faces": as above

"Regions": \( d \)-dim faces \( \leftrightarrow \) convex hulls of \( \mathbb{R}^d - \bigcup_i H_i \)

\[ r(\mathcal{A}) = \# \text{ regions} \]

\[ b(\mathcal{A}) = \# \text{ (relatively bounded) regions} \]

\[ \mathcal{A}_n : x_i = x_j \text{ for } 1 \leq i, j \leq n \text{ in } \mathbb{R}^n \]

To specify a region \( R \), I need to decide, for each \( c \in R \), whether \( c \cdot \alpha_i > 0 \) or \( c \cdot \alpha_j < 0 \).

So \( \mathcal{A}_n \) is \( n! \) and \( b(\mathcal{A}_n) = 0 \).

No surprise because \( V = \{e_i e_j \mid 1 \leq i, j \leq n\} \)

\[ A_n = H_n \]

\[ Z(V) = T_n \]

\( 0 \)-faces of \( T_n \) \( \leftrightarrow \) vertices of \( T_n \) \( \leftrightarrow \) permutations of \( S_n \)

\( n \)-faces of \( T_n \) \( \leftrightarrow \) faces of \( A_n \) \( \leftrightarrow \) ordered set partition of \( \{1, \ldots, n\} \)

(See: \( c_1 \subset c_2 \subset c_3 \subset c_4 \))