A max explicit explanation for Vol $T^n = n^{n-2}$:

**Zonotopes**

Def: A zonotope is a Minkowski sum of segments.

$1 + - + / + = \begin{array}{c}
\end{array}$

**Prop** The zonotope $Z = V_1 + \ldots + V_n \subset \mathbb{R}^d$ can be tiled into parallelepipeds:

$V_1 + \ldots + V_{d-1} = \sum_{i=1}^{d-1} V_i$

one for each basis $\{V_1, \ldots, V_{d-1}\}$ of $\mathbb{R}^d$.

**Cor** Vol $Z = \sum_{\{V_1, \ldots, V_{d-1}\}} |\text{det}(V_1, \ldots, V_{d-1})|$

$T_3 = \frac{1}{2} + \frac{1}{2} + \frac{3}{2} = \begin{array}{c}
\end{array}$

Prop: The volume of a parallelepiped per spanning tree $T$ all have volume $1$.

This is true for $T^n$ as well.

Sketch of proof: "Just induce!"

But to make this rigorous we need some machinery. $\Rightarrow$ hyperplane arrangement, etc.

The "dual" object can be easier to work with:

A hyperplane arrangement $A = \{H_i, \ldots, H_n\} \subset \mathbb{R}^d$ is a collection of hyperplanes $H_i = \{c \in \mathbb{R}^d: c \cdot v_i = 0\}$

**Ex:** $v_i \mapsto \begin{array}{c}
\end{array}$

hyp. arr. $\leftrightarrow$ zonotope

Def: A fan $\mathcal{F} = \{C_i, \ldots, C_n\}$ in $\mathbb{R}^d$ is a polyhedral complex of cones $C_i$. It is complete if $\cup C_i = \mathbb{R}^d$ and pointed if $\{0\} \in \mathcal{F}$.

**Ex 1** A hyp. arr. $\mathcal{F}$.

F pesos: $c \in \mathbb{R}^d$: $c \cdot v_i = 0$.

$\mathcal{F} = \{\text{fan of } A\}$

3 P polytope

For each face $F$,

$N_F = \{c \in \mathbb{R}^d: P_c \supset F\}$

= dual cone when $F$ is max.

Normal fan of $\mathcal{F}$:

$N(\mathcal{F}) = \{N_F: F \text{ fan of } \mathcal{F}\}$

**Prop** The normal fan of the zonotope $Z = V_1 + \ldots + V_n$ is the fan of the arrangement $H_i$: $v_i \cdot x = 0$

Proof: (For verification) $v$ index of $Z \Rightarrow v = v_1 + v_2 + v_3 + v_4$

$C_{\max} \text{ in } (Z = v_1 + v_2 + v_3 + v_4) \Rightarrow (C_{\max} \text{ in } v_1, v_2, v_3, v_4) \Rightarrow \text{face of } A$.

Theorem: A zonotope is orientable.