Lemmas (a<sub>i</sub> → a<sub>n</sub>) is a parking function if and only if all cars can park.

**Proof:** If everyone can park,

- The car at spot 1 can only park at 1, so it must park there.
- Suppose a car can't park. Let's say car 1 parks at spot 1, car 2 parks at spot 2, and so on, up to car k. Then car k+1 must park at spot k+1, and car k+2 must park at spot k+2, and so on.
- This implies that car k+1 parks at spot k+1, and car k+2 parks at spot k+2, and so on, up to car n.
- Thus, we have a parking function.

**Theorem:** There are (mn)<sup>n</sup> parking functions of length n.

Change the road: → 2<sup>3</sup> and let a car circle around until it can park. Allow O as a preference.

- Now there are (mn)<sup>n</sup> preferences possible.
- If (a<sub>i</sub> → a<sub>j</sub>) leaves spot k empty, then (a<sub>k</sub>, a<sub>k+1</sub>) ⇔ k<sub>th</sub> element (mod mn)
- (a<sub>n-k</sub> → a<sub>n-1</sub>) ⇔ k<sub>th</sub> element empty
- So exactly one of these leave spot 0 empty.
- So exactly one of them is a parking function (and hence no O's in the preference).

- So # of parking functions = (mn)<sup>n</sup> = (mn)<sup>n</sup>

By the way...

$$PF_3 = \begin{pmatrix} 123 \ 132 \ 231 \ 213 \ 321 \ 312 \end{pmatrix}$$

How many equivalence classes of parking functions?

Equivalently, how many sequences 1 ≤ j₁ ≤ j₂ ≤ ... ≤ jₙ ≤ n satisfy j<sub>i</sub> ≥ i (all i)?

2 3 5 7 7 9 9 → "Dyck path" 3 steps ↑ and 7 steps above diagonal

Let the number of Dyck paths of length n be dₙ.

$$dₙ = \frac{n}{k=1} \sum d_{k-1} d_{n-k}$$

Also, d₁ = 1

So dₙ = Cₙ = \frac{1}{mn} \binom{2n}{n}

The Catalan numbers!

- While we're at it...

**Theorem:** (Stasheff, Postnikov)

- The Minkowski sum

$$\sum_{(i,j) \in \Delta_{s}} \sum_{(i,j)}$$

is a realization of the associahedron.

Recall: The associahedron is a polytope with

- Cₙ vertices ↔ triangulations of (n-2)-gon
- faces ↔ subdivisions of (mn)-gon