From lattice points to facet:

**Theorem (Face relation)**

If $P$ is a $d$-polytope with $f_k$ $k$-faces,

$$f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^d f_d = 1$$

**Pf. (For $P$ lattice)**

Note: $P = (\cup \mathcal{F}^n)$, so

$$\mathcal{L}_P(t) = \sum_{\mathcal{F} \subseteq P} \mathcal{L}_{\mathcal{F}^n}(t) = \sum_{\mathcal{F} < P} (-1)^{\dim F} \mathcal{L}_F(-t)$$

Recall $[t^0] \mathcal{L}_P(t) = 1$ (well of $t^0$) so

$$1 = \sum_{\mathcal{F} < P} (-1)^{\dim F} f_F = \frac{d}{d-1} \sum_{k=0} (-1)^k f_k.$$  \[\Box\]

- In 3-D, facets of polytopes are classified.
- In $d$-dim, this seems hopeless. The "cd-index" (which encodes flags, not just facets) seems better than the facet.
- But for simple/simplicial polytopes, not hopeless!

**A linear relations**

Let the h-vector of $P$ be

$$h_k = f_k - (d-k) f_{k-1} + (d-k+2) f_{k-2} - \cdots + (-1)^k (d-k).$$

Stanley's trick:

```
\begin{array}{cccc}
1 & 6 & 11 & 5 \\
6 & 6 & 1 & \\
11 & 1 & & \\
5 & & & \\
\end{array}
```

$$h = 1 \ 3 \ 3 \ 1 \ 1$$

**Fact (Lee)**

Then we all the linear relations between the $f_k$.

**Pf.** This translates to

$$\sum_{i=0}^d (-1)^i (\binom{d}{i}) f_i = \sum_{i=0}^d (-1)^i \binom{d}{i} \mathcal{L}_F(-t)^i$$

which, after some manipulation, is equal to \[\Box \]

$$f_{\alpha \beta} = \sum_{i=0}^d (-1)^i \binom{d}{i} f_{\alpha \beta}.$$ (simple)

So let's prove (x) for a **simple** polytope $Q$.

**Note:** For each face,

$$\mathcal{L}_F(t) = \sum_{G \subseteq F} \mathcal{L}_G(t) = \sum_{G \subseteq F} (-1)^{\dim G} \mathcal{L}_G(-t)$$

So

$$\sum_{\mathcal{F} \subseteq \mathcal{Q}} \mathcal{L}_F(t) = \sum_{\mathcal{F} \subseteq \mathcal{Q}} \sum_{\mathcal{G} \subseteq \mathcal{F}} (-1)^{\dim G} \mathcal{L}_G(-t)$$

$$= \sum_{\mathcal{G} \subseteq \mathcal{Q}} (-1)^{\dim G} \mathcal{L}_G(-t) \sum_{\mathcal{F} \subseteq \mathcal{Q}} 1$$

Now just take coef of $t^0$ as above.

**B Non-linear conditions**

The "g-theorem" of Billera-Lee-Stanley completely classifies the $h$-vectors of simplicial polytopes.