From cones to polytopes.

\[ P = \text{conv} \{ v_1, \ldots, v_n \} \subseteq \mathbb{R}^d \]

\[ \Rightarrow \text{cone}(P) = \text{cone} \{ [v_1], \ldots, [v_n] \} \subseteq \mathbb{R}_+^n \]

Recall:

\[ L_P(t) = |tP \cap \mathbb{Z}^d| = \sigma_{L_P}(1) \]

\[ \text{Ehr}_P(z) = \sum_{t \geq 0} L_P(t) z^t \]

Check: \( \text{cone}(P) \cap \langle \text{hyperplane} \rangle = tP \), \( x_{\text{anh}} = t \)

So:

\[ \sigma_{\text{cone}(P)}(z_1, \ldots, z_d) = \sum_{m \in \text{cone}(P)} z^m \]

\[ = \sum_{m \in \text{cone}(P)} z^{m_1} \ldots z^{m_d} \]

\[ \sum_{m_1=0}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=2}^{\infty} \ldots z^m \]

So:

\[ \sigma_{\text{cone}(P)}(z_1, \ldots, z_d) = 1 + \sigma_P(z) z_{\text{anh}} + \sigma_{2P}(z) z_{\text{anh}}^2 + \ldots \]

Lemma:

Let \( f : \mathbb{N} \to \mathbb{C} \) be a function.

If \( f(n) \) is a polynomial of degree \( d \),

\[ \langle z \rangle \left( \sum_{n=0}^{\infty} f(n) z^n = \frac{g(z)}{(1-z)^{d+1}} \right) \]

where \( g \) is a polynomial of degree \( d \) with \( g(0) \neq 0 \)

Proof:

HW4.

Corollary:

\( L_P(n) \) is a polynomial in \( n \) of degree \( d \) for \( P \) a simplex.

Corollary:

\( L_P(n) \) is a polynomial in \( n \) of degree \( d \) for any polytope \( P \) with integer coefficients.

Proof: "Just triangulate." •

Also: The \( h_i \) above are \( \geq 0 \) "h*-vector" of \( P \).