How good is linear programming?
That is related to how long these paths get.

P polytope
\( d(P) = \text{diameter of } P \)
\( = \text{largest between two vertices of } P \)

let \( \Delta(d,n) = \text{max diameter of a chain polytope with } n \text{ facets} \)

How large is \( \Delta(d,n) \)? Is it polynomial in \( n \) and/or \( d \)?

**Hirsch Conjecture (1957)**
\[ \Delta(d,n) \leq n-d \]

**Theorem (Santos, June 2010)**
The Hirsch conjecture is false!
There is a 43-polytope \( P \) with 86 facets such that \( d(P) = 44 \).

Still, understanding \( \Delta(d,n) \) is very much open.

**Open**: \( \Delta(d,n) \leq \text{polynomial in } n,d \)?

**Step 1** (Klee-Walkup)
\[
(\Delta(d,n) \leq n-d) \quad \text{for all } n,d
\]
\[
\rightarrow \quad (\Delta(d,2d) \leq d) \quad \text{for all } d
\]

**Step 2**

If \( P \) is a spindle of dim \( d \), \( n \geq 2d \) facets, diam \( d \geq d \)
there is a spindle of dim \( d+1 \), \( n+1 \) facets, diam \( d+1 \geq d+1 \)

**Step 3**

There is a spindle of dim 5, 48 facets, diam 6

**Thm.**

There is a spindle of dim 43, 86 facets, diam 44

**Note** A trick allows us to see a 5-spindle on a 3-sphere, so "proof is in 3-D."