Lecture 7
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Two more constructions

1. **Pyramids**
   
   $P \subseteq \mathbb{R}^d$ polytope
   
   Embed $\mathbb{R}^d$ as $\{ x_{d+1} = 0 \} \hookrightarrow \mathbb{R}^{d+1}$
   
   $\text{Pyr}(P) = \text{conv}\left( \{(p^i)_0 : p \in P \} \cup \{ (0^i)_0 \} \right)$

   Face structure of $\text{Pyr}(P)$ is determined by the face structure of $P$. (HW)
2. Vertex Figures

$P \subset \mathbb{R}^d$

$v$ vertex, say $v = P_c$

$c \cdot v = c_0$

$c \cdot p \leq c_0 \quad p \in P$

Choose $c_1 < c_0$ (but close to $c_0$)

$P/v = P \cap \{ x : c \cdot x = c_1 \}$

Combining type of $P/v$ depends only on the combining type of $P, v$

$(k-1)$-faces of $P/v$ \leftrightarrow $k$-faces of $P$ containing $v$ (Ex.)
THE FACE LATTICE

Def: A poset (partially ordered set) \((P, \leq)\)

is a set \(P\) equipped with a binary relation \(\leq\)
such that:

- \(x \leq x\) \(\forall x \in P\)
- \(x \leq y, y \leq z \Rightarrow x \leq z\) \(\forall x, y, z \in P\)
- \(x \leq y, y \leq x \Rightarrow x = y\)

Ex. \((\mathbb{N}, \leq)\):

\[ B_n = \left(2^{[n]}, \subseteq\right) \quad \text{"Boolean poset"} \]

Hasse diagram:

- \([\text{element}]\)
- \([\text{cover rels}]\)

\(U \lor U < U\)
\(V \lor W \text{ s.t. } U < W < V\)
$P$ polytope
$L(P) = \text{"face poset"} = \left( \{\text{faces of } P\} , \leq \right)$

Ex. $L(\Delta_{d-1}) = B_d$

$\Delta_3 = b \xrightarrow{a} d$

$P, Q \text{ are combin. isom. if } L(P) \cong L(Q)$
$P$ poset

chain: $p_1 < p_2 < \ldots < p_k$

$P$ graded if for any $a < b$, all maximal chains from $a$ to $b$ have same length

(i.e., $P$ has "levels" or "ranks")

$B_n$ graded

$\text{rk}(S) = |S|$
A lattice is a partially ordered set (poset) in which any two elements have a unique least upper bound (join) and a unique greatest lower bound (meet). Formally:

- A lattice is a poset $(P, \leq)$ with a binary operation $\wedge$ (meet) and $\vee$ (join) such that for any $x, y \in P$:
  - $x \wedge y$ is the greatest lower bound of $x$ and $y$.
  - $x \vee y$ is the least upper bound of $x$ and $y$.

**Example:**

- $B_n$ is a lattice if:
  - $A \wedge B = A \cap B$
  - $A \vee B = A \cup B$

A diagram illustrates that $B_n$ is a lattice, whereas another diagram shows that the set $\{1, 2, 3, 4\}$ is not a lattice, as it does not satisfy the lattice properties.

**Note:** Lattices have a maximum $\top$ and a minimum $\bot$. 
Theorem  P polytope
(a) $L(P)$ is a lattice, graded by $\text{rk}(F) = \dim(F) + 1$
(b) Every interval $[F, G]$ is also a face lattice.
(c) Every interval of height 2 is a diamond.
(d) The "opposite poset" $L(P)_{op}$ is also a face lattice.

(a) $F \land G = F \cap G$
$F \lor G = \bigwedge(\text{upper bounds of } F, G)$

$(P \text{ point } \land \text{ has } 0, \hat{1}) \Rightarrow \text{ has } V$

Ex
b) \[ L(P) \quad L((G^\lambda)^*) \quad \rightarrow \quad a) \quad \text{Graded} \]

c) \[ L(G) \quad [f.g] : L(\Omega) = \diamond \]

d) \[ L(P)^{op} \cong \hat{\Phi}(P^\lambda) \quad \text{polar pol. of } P \quad \text{(next time)} \]