FACES OF POLYTOPES

$P \subseteq \mathbb{R}^d$ polytope, $c \in \mathbb{R}^d$

The face of $P$ in direction $c$ is

$P_c = \{ x \in P | c \cdot x \text{ is max} \}$

Note: $P$ is bounded $\rightarrow$ say

$P_c = \{ x \in P | c \cdot x = c_0 \}$

$\dim P_c = ?$

$\textbf{c} \cdot x = c_0 \text{ is the smallest valid slab for } P$
Remarks: affine spaces

An affine subspace of $\mathbb{R}^d$ is:
- $\{x \in \mathbb{R}^d : Ax = b\}$
- translate of a (vector) subspace
- affine span of a set $V \subset \mathbb{R}^d$,
  $\text{aff}(V) = \{\lambda_1 v_1 + \cdots + \lambda_r v_r : \sum \lambda_i = 1\}$

Say $U \{v_1, \ldots, v_k\}$ is affine independent if no $v_i$ is in affine span of $U - v_i$.

$\dim U = \text{size of largest affine independent set} - 1$

$(v_1, \ldots, v_k \text{ affine independent}) \iff (v_1, \ldots, v_k) \text{ linear independent}$
The dimension of a face $F$ of $P$ is $\dim(\text{aff}(F))$.

$\dim 0 =$ verts. $\quad$ codim 1 = facets
$\dim 1 =$ edges $\quad$ codim 2 = ridges

The f-vector of $P$ is $f(P) = (f_0, f_1, f_2, \ldots, f_d)$

where $f_i =$ # of $i$-faces of $P$

The f-polynomial of $P$ is $f_0x^0 + \cdots + f_dx^d = f_P(x)$

$$f_P(x) = 6 + 12x + 8x^2 + x^3$$

⚠️ Other authors have slightly different conventions
Ex  \[ C_d = \text{conv}(\{-1,1\}^d) = \{ x : -1 \leq x_i \leq 1 \} \quad C_3 \]

Goal: compute faces.

Let  \( v \in \mathbb{R}^d \), compute \((C_d)_v\)

\[(C_d)_v = \{ x \in [-1,1]^d : V_1 x_1 + \ldots + V_d x_d \leq \text{max} \} \]

If \( V_i > 0 \), \( V_i x_i \leq V_i \quad (x_i : 1) \)

\( V_i < 0 \), \( V_i x_i \leq -V_i \quad (x_i : -1) \)

\( V_i = 0 \), \( V_i x_i = 0 \quad \text{(any } x_i) \)

Say \( v = (0, 0, 0, 0, 0, 0, 0, 0) \)

\[(C_d)_v = \{ (1, -1, x, 1, x, x, x, x) \} \cong C_3 \]
(faces of $C_d$) $\leftrightarrow$ sign patterns $\{(+,-,0)^d\}$

\[
\dim \text{ face} \quad \leftrightarrow \quad \# \text{ of } 0s
\]

\[
f_k(C_d) = \binom{d}{k} 2^{d-k}
\]

\[
\text{por. of } 0s \quad \uparrow \quad \text{other \ por. } + \ or \ -
\]

\[
f_{C_d}(x) = \sum_{k=0}^{d} f_k x^k = \sum_{k=0}^{d} \binom{d}{k} 2^{d-k} x^k
\]

\[
= (x+2)^d
\]
There are several "obvious" things to prove.

Prop: \( P \text{ polytope } \Rightarrow P = \text{conv}(\text{vert}(P)) \)

1. Let \( P = \text{conv}(V) \)
   - If \( v \in V \) is sl.
     - \( v \in \text{conv}(V-v) = \text{conv}(v) \)
   - Keep dim. all superfluous members of \( V \) until it's no longer possible. Call that set \( W \)

Claim: \( W = \text{vert}(P) \)

2. Let \( v \in \text{vert}(P) \), assume \( v \notin W \Rightarrow v \in \text{conv}(W-v) \)
   - While \( v = \lambda_1 W_1 + \cdots + \lambda_k W_k \) with \( v \in W-v \lambda_i \geq 0 \)
     - Assume \( c \cdot v = c_0 \), \( c \cdot P \leq c_0 \) for \( P \cdot v \)

\( c \cdot v = \lambda_1 (c \cdot W_1) + \cdots + \lambda_k (c \cdot W_k) \) 
\( c_0 < c_0 / \lambda_1 + \cdots + \lambda_k = c_0 \)
\[ \leq: \text{Let } w \in W \quad w \in \text{conv} \left( \frac{W - w}{w} \right) \]

\[ w \in \text{conv} W' \Rightarrow \exists t > 0: w' = Wt, \ t \perp \Pi t \]

\[ \Rightarrow \exists t : \left( \frac{d}{w} \right)_t = (\cdot)'_t, \ t \geq 0 \]

\[ \Rightarrow \exists a: a\left( \frac{d}{w} \right)_t > 0, \ a\left( \cdot \right)_t < 0 \]

\[ \Rightarrow \exists (\beta, -\beta): \beta 1 - b \leq 0, \ \beta - bw < 0 \]

\[ \Rightarrow \exists \beta, b: bW < (\beta \cdot \beta'), \ bw > \beta \]

\[ \Rightarrow w \text{ maximizes } b \cdot x \Rightarrow w \in \text{vert } \Pi. \]
Other "obvious" facts about polytopes and their faces

Prop: \( P = \) polytope \( V = \) \( \text{vert}(P) \)

(i) Any face \( F \) of \( P \) is a polytope, \( \text{vert}(F) = \text{vert}(P) \cap F \)

(ii) \( F, G \) faces \( \Rightarrow F \cap G \) face

(iii) \( F \) face of \( P \), \( G \) face of \( F \) \( \Rightarrow G \) face of \( P \)

(iv) \( F \) face \( \Rightarrow F = P \cap \text{aff}(F) \)

(ii) \( F = P_b, G = P_c \Rightarrow F \cap G = P_{bc} \)

(iii) \( F = P_b, G = F_c \Rightarrow G = P_{b+c} \) (if small)