Remember:
\[ M_1 = (E_1, C_1) \rightarrow M_1 \oplus M_2 = (E_1 \cup E_2, C_1 \cup C_2) \] no circuits involving both \( E_1 \) and \( E_2 \)

So given a matroid \( M \), how do we "factor" it as a direct sum \( M = M_1 \oplus \cdots \oplus M_k \)?

**Def** Say \( a \sim j \) if \( a \) and \( j \) are in a circuit of \( M \)
**(Idea:** They are then in the same \( M_i \))

**Exercise** This is an equivalence relation:
\[ a \sim j, b \in C_1 \quad j, k \in C_2 \rightarrow D, G, U \sim -j \quad \text{need a strong elimination axiom to get } k \in D \]

If the equivalence classes are \( T_1, \ldots, T_k \) then
\[ M = (M_1 T_1) \oplus \cdots \oplus (M_k T_k) \]

This is the unique decomposition of \( M \) into connected components.

**Ex:** \( M(\triangledown) = M(\triangledown) \oplus M(\cdot) \oplus M(\circ) \)

**Def** If \[ P = \text{conv} (P_1, \ldots, P_m) \in \mathbb{R}^d \]
\[ Q = \text{conv} (Q_1, \ldots, Q_n) \in \mathbb{R}^P \]
\[ P \times Q = \text{conv} (P_i, Q_j, i \leq i \leq m, 1 \leq j \leq n) \in \mathbb{R}^{d+e} \]

**Ex:** \( P = \frac{1}{\sqrt{2}} \)
\[ Q = \frac{1}{\sqrt{2}} \]
\[ P \times Q = \frac{1}{\sqrt{2}} \]

**Prop** \( M = M_1 \oplus M_2 \Rightarrow P_M = P_{M_1} \times P_{M_2} \)
Prop. $\dim P_M = |E| - 1$ for $M$ connected (only $x_M = r(M)$)

Proof. Any $i$ and $j$ in a circuit $C$. Complete $C - j$ to a basis $B$.
$C$ is the basic circuit of $B$ with respect to $i \Rightarrow B \cup i - j \in B$
So any $e_i - e_j$ is an edge of $P_M$, and their span is $(|E| - 1) - \dim$.

**Theorem** $\dim P_M = |E| - (\# \text{conn. comp. of } M)$

**Proof** $M = \Theta N_i \Rightarrow \dim P_M = \sum_i \dim N_i = \sum_i (\dim N_i - 1) = |E| - c(M)$.

In other words, the only equalities are:

\[
\begin{align*}
X_1 + X_2 + X_3 + X_4 + X_5 &= 2 \\
X_1 &= 1 \\
X_6 &= 0
\end{align*}
\]

One per conn. comp.

This brings us to: Which inequalities are facets?

$P_M = \{ x \in \mathbb{R}^E \mid x_i \geq 0, \sum_{i \in E} x_i \leq r(F), \sum_{i \in E} x_i = r \}$

Note: $x_6 \geq 0$ redundant: $x_i = r - \sum_{j \in E_c} x_i \geq r - r(E-c) \geq 0.$

**Definition** A facet $F \in E$ is a "facet" of $M$ if $\sum_{i \in E} x_i \leq r(F)$ is a facet of $P_M$.

**Prop** $M$ connected $\Rightarrow$ facet $\Leftrightarrow M$ if $M$ connected

**Proof** The bases maximizing weight $\sum_{i \in F}$ are the bases of the matroid $F$

\[ (M_{\text{IF}}) \oplus (M_{\text{IF}}) \to \dim = 1|E| - c(M_{\text{IF}}) - c(M_{\text{IF}}). \]

Def. A cyclic flat is a flat which is a union of circuits.

**Prop** $F$ is a cyclic flat of $M$ $\Leftrightarrow$ $F$ flat of $H$ $\Leftrightarrow F$ flat of $H^*$

**Def** Facets are cyclic flats.