Theorem (Gelfand, Gourley, MacPherson, Serganova, 1997)

Let $\mathcal{B}$ be a collection of k-sets of $E$.
Let $P_{\mathcal{B}} = \text{conv} \{ V_B : B \in \mathcal{B} \}$

$(E, \mathcal{B})$ is a matroid $\iff$ every edge of $P_{\mathcal{B}}$

is of the form $e_i - e_j$.

$\Rightarrow$ let $V_A, V_B$ form an edge. Then $A, B$ are the (only)
w-max bases for some weight vector.

Let $a \in A - B$. By symmetric exchange find $b \in B - A$ with

$A - a \cup b, B - b \cup a \in \mathcal{B}$

Since $w(A - a \cup b) + w(B - b \cup a) = \max w(A) + w(B)$, the base

$A - a \cup b, B - b \cup a$ must also be maximum. $\Rightarrow A - a \cup b = B$

$V_A - V_B = V_A - (V_A - a \cup b) = e_a - e_b$.

$\Leftarrow$ let $V_A, V_B$ be vertices of $P_{\mathcal{B}}$

$V_B - V_A = \sum \alpha_i \cdot E_i$

$\Rightarrow$ edges coming out of $V_A$.

Assume $V_A = 111000110000$ w/o $a$

$V_B = 000111110000$

$V_B - V_A = -111111100000$

$W \longleftrightarrow X \longleftrightarrow Y \longleftrightarrow Z$

Suppose $E_i = e_Y - e_Z$ occurs. $r \in A \rightarrow r \in X \cup Z$

$s \in A \rightarrow s \in W \cup Y$

If $r \in \mathbb{Z}$, $(V_B - V_A)_r > 0 \rightarrow r \in X$ Similarly $s \in W$
Now let's prove the basic exchange axiom.

Let $a \in A - B = W$. Since $V_B - V_A = \sum \alpha_i E_i$, some $E_i$ has a coordinate is $-1$.

Say $E_i = e_b - e_a$

$\Rightarrow V_A + E_i = V_{A \cup u - a}$ is a vertex of $P_B$

$\Rightarrow A \cup u \in E(B)$

Note: A way to build $P_B$ is to consider the standard simplex with vertices $e_1, e_2, \ldots, e_B$, and put a vertex on the barycenter of face $B$, which is $\frac{1}{r} (\sum_{b \in B} e_b) = \frac{1}{r} V_B$.

($B$ is a matroid) $\iff$ (edges of $P_B$ $\parallel$ edges of simplex)

$B = \{12, 13, 14, 23, 24\}$

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\begin{itemize}
  \item $\bigcirc$ a matroid
  \item $\times$ not a matroid
\end{itemize}