Claim. If G contains no routing from \{e,f\} to B_0, then there is a "bottleneck vertex" b such that any path from e or f to B_0 passes through b.

Proof. Consider a counterexample with the minimum number of edges possible.

Let \( \overrightarrow{e \rightarrow f} \) be an edge and \( H = G - (\overrightarrow{e \rightarrow f}) \). Then there is still no routing from e,f to B_0 in H. So let b be a bottleneck from \{e,f\} to B_0 in H. \( \Rightarrow \{b, v, v_3\} \) bottleneck for \{e,f\} \rightarrow B_0 in the graph G.

If I had a routing \{e,f\} \rightarrow \{v, b\} and a routing \{b, v\} \rightarrow B_0, I would get a routing \{e,f\} \rightarrow B_0.

So either

- I have no routing \{e,f\} \rightarrow \{v, b\} \Rightarrow bottleneck from \{e,f\} \rightarrow \{v, b\}

or

- I have no routing \{b, v\} \rightarrow B_0 \Rightarrow bottleneck from \{b, v_3\} \rightarrow B_0.

Note. More generally:

Hengen's Theorem. G=(V,E) directed graph, \( \forall A, B \in V:\)

\[
\frac{\text{Size of largest routing from } A \text{ to } B}{\text{Size of smallest bottleneck from } A \text{ to } B} = \frac{\text{Size of smallest bottleneck from } A \text{ to } B}{\text{Size of largest routing from } A \text{ to } B}
\]

Pf. Same.