Prove that a finite lattice \( L \) is semimodular iff whenever \( x \) and \( y \) both cover \( x \wedge y \), then \( x \wedge y \) covers both \( x \) and \( y \).

\[ \implies \]
Suppose \( L \) is semimodular, and \( x \) and \( y \) both cover \( x \wedge y \). Then
\[
 r(x) = r(y) = r(x \wedge y) + 1. 
\]
By semimodularity,
\[
 r(x) + r(y) = r(x \wedge y) + r(x \wedge y) + 1, 
\]
so
\[
 r(x \wedge y) + 1 + r(x \wedge y) + 1 > r(x \wedge y) + r(x \wedge y) \implies r(x \wedge y) + r(x \wedge y) \leq 2. 
\]
If \( x \wedge y \) didn't cover \( x \) and \( y \), then \( x \wedge y \) would have to be more than 2 steps higher than \( x \wedge y \).

\[ \impliedby \]
Suppose the cover relation holds. We will show that \( L \) is graded and that
\[
 r(x) + r(y) = r(x \wedge y) + r(x \wedge y). 
\]

\( L \) is graded: Suppose \( L \) is not graded. Therefore there are elements \( x \) and \( y \) in \( L \) between whom exist chains of different length. Since \( L \) is finite, let \( x \) and \( y \) be minimal with this condition — i.e., the interval \([x, y]\) is of minimal length. Let \( a \) and \( b \) be two chains of differing lengths from \( x \) to \( y \). Let \( x \wedge a, x \wedge b \) be covers of \( x \). In particular, by minimality assumption, \( x \wedge a \) and \( x \wedge b \) be covers of \( x \). Then by hypothesis \( x \), \( x \wedge a \) covers both \( x \) and \( x \wedge a \).
Now observe that \([x_1, y]\) and \([x_2, y]\) are both strictly shorter chains than \([x, y]\), hence by our minimality assumption these two intervals must be joined.

In particular, the chain along 1 from \(x_1\) to \(y\) — call it \(a'\) — must be the same length as the chain from \(x_1\) through \(x_1x_2\) up to \(y\). Since \(x_1x_2\) covers \(x_1, x_2\) up to \(y\) and finally this must be the same length as the chain from \(x_2\) through \(x_2y\) up to \(y\). And finally this will be the same length as the chain \(b'\) of \(x_2\) to \(y\) along \(b\). So \(a'\) and \(b'\) have the same length, and hence \(a\) and \(b\) also have the same length. Contradiction!

\(L\) satisfies the submodularity inequality.

Suppose it doesn’t, i.e., suppose that \(\exists x, y \in L\) such that

\[
\rho(x) + \rho(y) < \rho(x'v'y) + \rho(x'y) + \rho(x'vy).
\]

Pick \(x\) and \(y\) such that the length of the chain from \(xay\) to \(x'vy\) is minimal, and that \(\rho(x) + \rho(y)\) is minimal.

If \(x\) and \(y\) cover \(xay\) then the result swiftly follows by our cover relation assumption. So assume \(x\) and \(y\) don’t cover \(xay\). So without loss of generality let’s take \(x'\) between \(x\) and \(xay\). By minimality of \(x\) and \(y\),

we must have

\[
\rho(x') + \rho(y) > \rho(x'vy) + \rho(x'vy).
\]

Let’s add these two inequalities:

\[
(\rho(x) + \rho(y)) + (\rho(x'vy) + \rho(x'vy)) < (\rho(x'vy) + \rho(x'vy)) + (\rho(x) + \rho(y)).
\]

Since \(x'vy = x'vy\), we can simplify this inequality to

\[
\rho(x) + \rho(x'vy) < \rho(x'vy).
\]

we can see by the picture that \(xay(x'vy) > x\) and \(xay(x'vy) = x'vy\).

To make things simpler, let \(a = x\) and \(b = (x'vy)\). Then using the previous inequality we get

\[
\rho(a) + \rho(b) < \rho(a) + \rho(a'),
\]

But \(a'\) and \(ab\) are closer together than \(xvy\) and \(xay\), so we’ve contradicted the minimality of the chain from \(xay\) to \(x'vy\). This completes the proof!