Suppose that \( r : 2^E \rightarrow \mathbb{N} \) satisfies the standard axioms R1, R2, R3 for a rank function. Then R1' for \( r \) follows from R1 with \( X = \emptyset \). The lower bound \( r(A \cup \{a\}) - r(A) \geq 0 \) in R2' follows from R2 with \( Y = A \cup \{a\}, \ X = A \); the upper bound \( r(A \cup \{a\}) - r(A) \leq 1 \) is a consequence of R3 with \( X = A, Y = \{a\}, \) bounding the \( r(Y) \) and \( r(\emptyset) \) that then appear by R1. Finally, R3' comes from R3 in the situation \( X = A \cup \{a\}, \ Y = A \cup \{b\} \), where the bound from R3 is \( r(A \cup \{a\} \cup \{b\}) \leq r(A) \), which is equality by R2.

Conversely suppose that \( r : 2^E \rightarrow \mathbb{N} \) satisfies our local axioms R1', R2', R3'. We'll show R1, R2, R3 using telescoping sum techniques. To show R1, the lower bound \( 0 \leq r(X) \) is a trivial consequence of the range we've defined \( r \) with. For the upper bound, label the elements of \( X \) as \( x_1, \ldots, x_n \), and write \( X_k = \{x_1, \ldots, x_k\} \) for \( 0 \leq k \leq n \); then by R1' and R2', we have

\[
r(X) = r(X_n) = r(X_0) + \sum_{k=1}^{n} r(X_k) - r(X_{k-1}) \leq 0 + \sum_{k=1}^{n} 1 = n,
\]

which is R1.

For R2 we do similarly. Given \( X \) and \( Y \supseteq X \), write \( Y = X \cap \{y_1, \ldots, y_n\} \), and put \( Y_k = X \cap \{y_1, \ldots, y_k\} \). Then by R2',

\[
r(Y) - r(X) = r(Y_k) - r(Y_0) = \sum_{k=1}^{n} r(Y_k) - r(Y_{k-1}) \geq \sum_{k=1}^{n} 0 = 0,
\]

which is R2.

Finally, for R3, we first note that R3' can be restated, assuming R2', to assert that

\[
r(A \cup \{a\}) + r(A \cup \{b\}) - r(A) - r(A \cup \{a, b\}) \geq 0.
\]

For if either \( r(A \cup \{a\}) - r(A) \) or \( r(A \cup \{b\}) - r(A) \) is 1, then this condition is vacuous in light of R2'; but if \( r(A \cup \{a\}) - r(A) = r(A \cup \{b\}) - r(A) = 0 \), this condition is equivalent to R3'. Now given sets \( X \) and \( Y \), put \( X \setminus Y = \{x_1, \ldots, x_n\} \), \( Y \setminus X = \{y_1, \ldots, y_m\} \), and \( Z_{k,l} := (X \cap Y) \cup \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_l\} \)
\{y_1, \ldots, y_l\}. Then, using R3',

\[
\begin{align*}
& r(X) + r(Y) - r(X \cap Y) - r(X \cup Y) \\
& = r(Z_{n,0}) + r(Z_{0,m}) - r(Z_{0,0}) - r(Z_{n,m}) \\
& = \sum_{k=1}^{n} (r(Z_{k,0}) - r(Z_{k-1,0})) - (r(Z_{k,m}) - r(Z_{k-1,m})) \\
& = \sum_{k=1}^{n} \sum_{l=1}^{m} r(Z_{k,l}) - r(Z_{k,l-1}) - r(Z_{k-1,l}) + r(Z_{k-1,l-1}) \\
& \geq \sum_{k,l} 0 = 0,
\end{align*}
\]

which is R3.