homework five

Note. You are encouraged to work together on the homework, but you must state who you worked with. You must write your solutions independently and in your own words.

Nota. Los estudiantes en Bogotá pueden escoger cuatro de los cinco problemas. El quinto vale puntos adicionales.

1. Representability of $U_{2,n}$. Over which fields is the uniform matroid $U_{2,n}$ representable?

2. Representability over a field of characteristic zero has infinitely many forbidden minors. Let $p > 2$ be a prime number and let $\mathbb{F}_p$ be the field of $p$ elements. Let $L_p$ be the vector matroid of the following configuration of $2(p+1)$ vectors in $\mathbb{F}_p^{p+1}$:

$$V_p = \{(1,0,\ldots,0,0),(0,1,\ldots,0,0),\ldots,(0,0,\ldots,1,0),(0,0,\ldots,0,1), (0,1,\ldots,1,1),(1,0,\ldots,1,1),\ldots,(1,1,\ldots,0,1),(1,1,\ldots,1,0)\}$$

(a) Prove that $L_p$ is not representable over any field of characteristic 0.
(b) Prove that $L_p$ is not a minor of $L_q$ for $p < q$.

3. (optional) The non-Desargues matroid is not algebraic. Prove it.

4. Computing Möbius functions. Let $P$ and $Q$ be posets. Let $P \times Q$ be the poset whose elements are the pairs $(p,q)$ such that $p \in P$ and $q \in Q$, and whose order relation is given by:

$$(p_1,q_1) \leq_{P \times Q} (p_2,q_2) \quad \text{if and only if} \quad p_1 \leq_P p_2 \quad \text{and} \quad q_1 \leq_Q q_2.$$ 

(a) Prove that the Möbius function of $P \times Q$ is given by

$$\mu_{P \times Q}(p,q) = \mu_P(p)\mu_Q(q).$$

(b) Find the Möbius function of the Boolean lattice $2[^n]$.
(c) Find the Möbius function of the lattice $D_n$ of divisors of $n$.
(d) Find the Möbius function of the partition lattice $\Pi_n$.

5. Counting proper colorings and acyclic orientations of graphs. Let $G$ be a graph and $M$ be its graphical matroid. Let $c$ be the number of connected components of $G$.

(a) Let $q$ be a positive integer. Prove that $q^c \chi_M(q)$ is the number of ways of coloring the vertices of $G$ with $q$ given colors in such a way that neighboring vertices have different colors.

(b) Prove that $|\chi_M(-1)|$ is the number of ways of orienting each edge of $G$ in such a way that no directed cycles are formed.