homework four

Note. You are encouraged to work together on the homework, but you must state who you worked with. You must write your solutions independently and in your own words.

1. The partition lattice. A partition of \([n]\) is a collection \(\pi = \{S_1, \ldots, S_k\}\) of pairwise disjoint subsets of \([n]\) (called the blocks of \(\pi\)) whose union is \([n]\). Consider the set of partitions of \([n]\), with the following partial order: If \(\pi_1\) and \(\pi_2\) are partitions of \([n]\), say that \(\pi_1 \leq \pi_2\) if every block of \(\pi_2\) is a union of blocks of \(\pi_1\).
   (a) Prove that this defines a poset \(\Pi_n\).
   (b) Prove that \(\Pi_n\) is a lattice.
   (c) Prove that \(\Pi_n\) is graded, and describe its rank function.
   (d) Prove that \(\Pi_n\) is semimodular.
   (e) Prove that \(\Pi_n\) is atomic.
   (f) Prove that \(\Pi_n\) is the lattice of flats of \(M(K_n)\), the graphical matroid of the complete graph \(K_n\).

Note. Solving part (f) would immediately solve the other ones, since the lattice of flats of a matroid is geometric. However, the purpose of this exercise is to get your hands dirty and really get acquainted with the partition lattice, so I want you to solve (a)-(e) directly from the definitions.

2. Semimodular lattices. Prove that a finite lattice \(L\) is semimodular if and only if it satisfies the following condition:
   If \(x, y \in L\) are such that \(x\) and \(y\) both cover \(x \land y\), then \(x \lor y\) covers both \(x\) and \(y\).

3. Minors and duals. Let \(M\) be a matroid on \(E\) and let \(A \subseteq E\). Show the following:
   (a) \((M/A)^* = M^* \setminus A\)
   (b) \(cl_{M/A}(X) = cl_M(X \cup A) - A\) for all \(X \subseteq E - A\).
   (c) \(M/A\) has no loops if and only if \(A\) is a flat of \(M\).

4. Parallel elements in cotransversal matroids. Show that if \(e\) and \(f\) are parallel elements in a cotransversal matroid \(M\), then \(M \setminus e\) is also cotransversal.

5. The matroid of bases of minimum weight. Let \(M = (E, B)\) be a matroid and let \(w : E \to \mathbb{R}\) be a weight function on \(E\). For each real number \(r\), let \(E_r = \{e \in E \mid w(e) \leq r\}\). Notice that there are only finitely many different sets \(E_r\); let’s call them \(S_1, \ldots, S_k\).
   Let \(M_w\) be the matroid of bases of minimum weight of \(M\). Find a description of \(M_w\) in terms of the matroid \(M\), the sets \(S_i\), direct sums, and minors.