Permutations, revisited

Counting the n! perms. of \([n]\) was easy.
We now want more refined counts in terms of various "statistics", e.g., number of cycles, inversions, descents, etc.

Recall various notations for a permutation of \([n]\):
- 2-line: \[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 2 & 6 & 1 & 8 & 4 & 7 & 5
\end{pmatrix}
\]
- 1-line: 32618475
- cycle: \[(1\ 3\ 6\ 4)\ (2\ 5\ 8)\ (7)\]
- digraph:

The cycle notation is not unique. It is useful to choose a:
Standard representation: *in each cycle, largest # is first
  * cycles listed by increasing order of largest #

\[(1\ 3\ 6\ 4)\ (2\ 5\ 8)\ (7) = (2\ 6\ 4\ 1\ 3\ 7\ 8)\]

Pf. Several, see book.
Easiest by induction.

\[
X(x^1)(x^2)\cdots(x^{n-1}) = \sum_{k=0}^{n} \binom{n}{k} c(n,k) x^k
\]
The cycle of \( \omega_n \) is \( \tau = (g_1, \ldots, g_n) \) when \( n \) is odd.

**Prop.** There are \( \frac{n(n-1)}{2} \) cycles of length \( 2 \).

**Proof.** Suppose \( \omega \) is a cycle. To make an even number of pairs of the \( (g_i, g_j) \) when \( n \) is odd.

Let \( \tau = (g_1, \ldots, g_n) \) be the cycle of length \( n \).

- **Goal:** Enumerating \( \omega_n \) by inversions.

- **Inversion:**
  - In \( (w_1, w_2, \ldots, w_n) \) an inversion is a pair \( (w_i, w_j) \) s.t. \( w_i < w_j \) in sorted order. The number of inversions in \( \omega_n \) is \( \frac{n(n-1)}{2} \).

- **Prop.:** An inversion of \( \omega_n \) is a bijection.

- **Claim:** The fundamental bijection:
  - If \( w \) has \( k \) cycles, \( w \) has \( k \) "records."

\[
\sum_{n=1}^{\infty} x^n = \exp \left( \frac{x}{1-x} \right) = \frac{1}{1-x}
\]
The inversion table of $w \in S_n$ is $(a_1, \ldots, a_n)$
when $a_i = \# \{ j : (j,i) \text{ is an inversion of } w \}$

**Note:** $\text{Inv}(w) = a_1 + \cdots + a_n$ 
$a_i \leq n-i$

**Ex:** $2, 4, 1, 6, 3, 5 \rightarrow (2,0,2,0,1,0) = I(w)$

**Prop** The map $w \mapsto I(w)$ is a bijection.

$S_n \rightarrow \{(a_1, \ldots, a_n) : 0 \leq a_i \leq n-i \}$

**Pf** It suffices to construct the inverse map.

Given $(a_1, \ldots, a_n)$, build words $w^n, w^{n-1}, \ldots, w^2, w^1$ where $w^i$ is obtained from $w^{i+1}$ by inverting $i$ so that $a_i (\leq n-i)$ numbers to its left.

$(2,0,2,0,1,0) : w^6 = 6$ 
$w^5 = 65$ 
$w^4 = 465$ 
$w^3 = 4635$ 
$w^2 = 24635$ 
$w^1 = 241635$

$a_1 = 2$
$a_2 = 0$
$a_3 = 2$
$a_4 = 0$
$a_5 = 1$
$a_6 = 0$

Note that after $w^i$, the value of $a_i$ stays put,
so $I(w) = (a_1, \ldots, a_n)$.

**Cor.** \[ \sum_{w \in S_n} q^{\text{inv}(w)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}) \]

**Pf.** \[ \sum_{w \in S_n} q^{\text{inv}(w)} = \sum_{a_0, a_1, \ldots, a_n} \frac{q^{a_0} q^{a_1+\cdots+a_n}}{a_0! a_1! \cdots a_n!} = \frac{1}{(q-1)(q^2-1) \cdots (q^n-1)} \]

**Remark:** "$q$-analog"s

- $[n]_q = \frac{q^n-1}{q-1}$ is the "$q$-analog" of $\mathbb{N}$
- $(q)_n = (1+q+\cdots+q^{n-1}) = [n]_q!$ is the "$q$-analog" of $n!$

A "$q$-analog" of a combinatorial object is an object depending on $q$ which "reduces" to the object when $q=1$. Very nice but very common, useful.

Very common: $q$ prime power

For finite field of $q$ elements:

- Vector space $\mathbb{F}_q^n$ is the $q$-analog of $\{1, \ldots, q^n\}$

\[ \# \{ \emptyset \subseteq S_1 \subseteq \cdots \subseteq S_n = \{n\} \} = n! \]

\[ \# \{ \emptyset \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{F}_q^n \} = [n]_q! \]