Two basic counting principles:

- **Multiplication principle:**
  If there are \(a\) ways of performing task \(A\) and \(b\) ways of performing task \(B\) (regardless of the outcome of \(A\)), then there are \(ab\) ways of performing both tasks.

- **Addition principle:**
  If there are \(a\) ways of performing task \(A\) and \(b\) ways of performing task \(B\), then there are \(a+b\) ways of performing one of \(A\) or \(B\).

**Prop:** \(|2^S| = 2^n|\)

**Proof:** Let \(S = \{a_1, \ldots, a_n\}\). Choosing a subset \(T \subseteq S\) is the same as choosing:

- Does \(a_1 \in T\)? (2 outcomes)
- Does \(a_2 \notin T\)? (2 outcomes)
- Does \(a_3 \in T\)? (2 outcomes)
- 
  So I have \(2 \times 2 \times \ldots \times 2 = 2^n\) outcomes.

**Note:** This gives a bijection

\[\text{[subset of } S]\leftrightarrow [\text{seqs } (E_i : 1 \leq i \leq n): E_i = 0 \text{ or } 1]\]

Define \(\binom{n}{k} := |(S_k)|\) for \(|S| = n\)

\[= \# \text{ of ways of choosing } k \text{ elts from a set of } n \text{ elts}\]
Prop \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

If we want \( \binom{n}{k} \) wrong:

To choose a subset \( \{b_1, \ldots, b_k\} \) of \( \{a_1, \ldots, a_n\} \):

- Choose \( b_1 \) \((n \text{ choices})\)
- Choose \( b_2 \) \((n-1 \text{ all but } b_1)\)
  
  \[ \vdots \]
- Choose \( b_k \) \((n-(k-1) \text{ all but } b_1, b_2, \ldots, b_{k-1})\)

Total choices: \( n(n-1) \cdots (n-k+1) \).

But we wanted ordered \( k \)-sets. We showed

\[
\left( \begin{array}{c}
\text{# ordered } k\text{-sets} \\
\text{of } \{a_1, \ldots, a_n\}
\end{array} \right) = \frac{n!}{k!(n-k)!} \tag{9}
\]

Now want them differently. To choose an ordered \( k \)-set of \( \{a_1, \ldots, a_n\} \):

- Choose an unordered \( k \)-set \( \binom{n}{k} \) choices.
- Order it \( \binom{n}{k} k! \) choices.

So

\[
\left( \begin{array}{c}
\text{# ordered } k\text{-sets} \\
\text{of } \{a_1, \ldots, a_n\}
\end{array} \right) = \binom{n}{k} k! \tag{x}\]

We wanted the same quantity in two (correct) ways, so combining (9), (x) we get

\[
\frac{n!}{k!(n-k)!} = \binom{n}{k} k! \tag{10}
\]

Subject and GFs:

The multivariate GF for subsets of \( \{w=x_1, x_2, \ldots \} \) is

\[
\sum_{A \subseteq \{w\}} \prod_{a \in A} x_a = (1+x_1)(1+x_2)\ldots(1+x_n)
\]

by inspection or by induction.

Now let \( x_1 = \ldots = x_n = x \):

\[
\sum_{A \subseteq \{w\}} x^{\left| A \right|} = (1+x)^n
\]

Binomial theorem

\[
\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n
\]

Note: \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \) from combinatorics.

Exercise: \( \sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1} \) from combinatorics.

Composition:

A composition of \( n \) is a way of expressing \( n \) as an ordered sum of positive integers

\[ \begin{array}{c}
1+1+1 \\
2+1+1 \\
1+2+1
\end{array} \]

A \( k \)-comp is one with \( k \) parts.

A weak comp is one into non-negative integers.
Prop. There are \(2^{n-1}\) compositions of \(n\)

\[
(\text{eq}) \quad \text{\(k\)-comps of \(n\)}
\]

**Proof:** To choose a comp of \(n\):

- Write \(n = 1 + 1 + 1 + \cdots + 1 + 1\)
- Delete some of the \(1\)'s \((2^{n-1} \text{ choices})\)
- Group consecutive 1's into one part

\[(\text{Ex:} \quad 8 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \quad \Rightarrow \quad 8 = 1 + 3 + 2 + 2)\]

Similarly for \(k\)-comps. \(\square\)

**Prop.** There are \((2^{n-k-1})\) \(k\)-comps of \(n\)

**Proof:** \(n = a_1 + \cdots + a_k\) is a \(k\)-comp of \(n\) \(\Leftrightarrow\)

\[n - k = (a_1 + 1) + \cdots + (a_k + 1)\] is a \(k\)-comp of \(n + k\). \(\square\)

Multisets

A \(k\)-multiset is a set with possibly repeated elements, like \(\{1, 2, 2, 2, 4, 5, 5, 7\} = \{1, 2^3, 4, 5^2, 7\}\)

This multiset has cardinality/size 8.

Let \((\{S\}) = \{k\text{-multisets on } S\}\). Let \([S] = |\{S\}|\)

Let \(n!\) choose for \(1\leq k \leq n\). \(\binom{n}{k} = \frac{n!}{k!(n-k)!}\)

Prop. \((\binom{n}{k}) = \binom{n+k-1}{k}\)

**Proof:** If a \(k\)-multiset \(S\) on \([n]\) has \(q\) copies of \(1\) \((1\leq q \leq n)\) then \(a_1 + \cdots + a_k = k\) is a weak \(n\)-composition of \(k\), and vice versa. So

\[
\binom{n}{k} = \binom{k+n-1}{k-1} = \binom{n+k-1}{k} \quad \square
\]

Multisects and GFs

The multivariate GF for multisets on \([n]\) is

\[
\sum_{v: [n] \rightarrow \mathbb{N}} x_v^n = (1 + x_1 x_2 + \cdots)(1 + x_1 x_2 + \cdots) \cdots (1 + x_1 x_2 + \cdots)
\]

Again letting \(x_1 = \cdots = x_n = x\),

\[
\sum_{v: [n] \rightarrow \mathbb{N}} x_v^{v(1) + \cdots + v(n)} = (1 + x_1 x_2 + \cdots)^n
\]

\[
\sum_{S \text{ multiset on } [n]} x^{1\text{mult}} = (\frac{1}{1-x})^n
\]

Linear in \(x\), binomial theorem

\[
\sum_{k=0}^{\infty} (\binom{n}{k}) x^k = (1-x)^{-n} = \sum_{k=0}^{\infty} (-n)^{-n} (-x)^k
\]

So

\[
(\binom{n}{k}) = (-1)^k \binom{-n}{k}
\]
The multinomial coefficient \((a_1, \ldots, a_m)^n\) is the number of ways of splitting an \(n\)-set into an \(a_1\)-set, an \(a_2\)-set, \ldots, and an \(a_m\)-set in order (where \(a_1 + \cdots + a_m = n\)).

For example, \(\binom{n}{k, n-k} = \binom{n}{k}\).

Prop: The number of permutations of \(\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}\}\)

is \(\binom{n}{a_1, \ldots, a_m}\) where \(a_1 + \cdots + a_m = n\).

Pick out of the \(a_1 + \cdots + a_m\) positions. 
Choose which \(a_i\) of them will hold the 1s.
\[
\begin{align*}
a_1 & \quad 25 \\
\vdots & \quad \\
a_m & \quad 
\end{align*}
\]

Prop: \(\binom{n}{a_1, \ldots, a_m} = \frac{n!}{a_1! \cdots a_m!}\)

Prop: \((x_1 + \cdots + x_m)^n = \sum_{a_1 + \cdots + a_m = n} \binom{n}{a_1, \ldots, a_m} x_1^{a_1} \cdots x_m^{a_m}\)

Prop: In the \(m\)-dim. box of dimensions \(a_1 \times \cdots \times a_m\), there are \((a_1, \ldots, a_m)\) shortest lattice paths from \((a_1, 0)\) to \((a_1, a_m)\).