4. (Compositions of compositions) Let \( n \) be a fixed positive integer. Find the number of ways of choosing a composition \( \alpha \) of \( n \), and then choosing a composition of each part of \( \alpha \).

Suppose \( n = a_1 + a_2 + \cdots + a_k \) is a composition obtained in that way, so let’s count from how many compositions it may have come. That problem is equivalent to divide the sum in groups, and that can be obtained choosing a subset of the \( k - 1 \) plus signs in the sum, that can be done in \( 2^{k-1} \) ways for a composition of length \( k \), an how many compositions of length \( k \) are there? is like having \( n = 1 + 1 + \cdots + 1 \) and choosing \( k - 1 \) plus signs from the \( n - 1 \) available, and that can be done in \( \binom{n-1}{k-1} \) ways.

So our desired result is

\[
\sum_{k=1}^{n} \binom{n-1}{k-1} 2^{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^k = (1 + 2)^{n-1} = 3^{n-1}
\]

5. (Compositions with bounded parts) Let \( k \) be a fixed positive integer. For each \( n \) let \( c_k(n) \) be the number of compositions of \( n \) such that every part is less than \( k \). Prove that

\[
\sum_{n \geq 0} c_k(n) x^n = \frac{1 - x}{1 - 2x + x^k}
\]

We have the recurrence relation \( c_k(n) = c_k(n-1) + c_k(n-2) + \cdots + c_k(n-(k-1)) \) for \( n > 0 \) (if we suppose that if \( n < 0 \) then \( c_k(n) = 0 \)) depending of which number is the first in the composition.

So \( C(x) = \sum_{n \geq 0} c_k(n) x^n = c_k(0) + c_k(1) x + c_k(2) x^2 + \cdots + c_k(k-1) x^{k-1} + c_k(k) x^k + \cdots \)

\[= c_k(0) + c_k(0) x + (c_k(1) + c_k(0)) x^2 + \cdots + (c_k(k-1) + c_k(k-2) + \cdots + c_k(1)) x^k + \cdots \]

\[\Rightarrow C(x) = 1 + C(x)(x + x^2 + \cdots + x^{k-1}) \]

\[\Rightarrow C(x)(1 - x - x^2 - \cdots - x^{k-1}) = 1 \]

\[\Rightarrow C(x) = \frac{1}{1 - x - x^2 - \cdots - x^{k-1}} = \frac{1 - x}{(1-x)(1-x^2 - \cdots - x^{k-1})} = \frac{1 - x}{1 - 2x + x^k} \]

6. (Bonus problem: NE-path tilings.) Consider an \( n \times n \) square grid. An NE-path is a sequence of one or more unit squares such that each unit square is either directly above or directly to the right of the previous unit square. Find the number of ways of tiling the \( n \times n \) square grid with NE-paths.

First let’s divide the grid in half by a diagonal because the problem is symmetric, so let’s just consider the bottom left middle. And let’s look at the grid by it diagonals like this

\[
\begin{array}{cccc}
 4 & & & \\
3 & 4 & & \\
2 & 3 & 4 & \\
1 & 2 & 3 & 4
\end{array}
\]

So a NE-path tiling will induce some arrows that will represent how are the connections of each path, having no arrows meaning the path ends there, always given by going from