3. (Compositions of compositions) Let \( \alpha \) denote the smallest number among \( n_1, \ldots, n_k \).

Before we begin our proof let’s quickly note that if at least one \( n_i = 0 \), then \( \min(n_1, \ldots, n_k) = 0 \).

Let

\[
A(x_1, \ldots, x_k) = \sum_{n_1, \ldots, n_k \geq 0} \min(n_1, \ldots, n_k)x_1^{n_1} \cdots x_k^{n_k}
\]

Then we have that

\[
(1 - x_1x_2 \cdots x_k)A(x_1, \ldots, x_k) = A(x_1, \ldots, x_k) - x_1x_2 \cdots x_k A(x_1, \ldots, x_k)
\]

\[
\sum_{n_1, \ldots, n_k \geq 0} \min(n_1, \ldots, n_k)x_1^{n_1} \cdots x_k^{n_k} - \sum_{n_1, \ldots, n_k \geq 0} \min(n_1, \ldots, n_k)x_1^{n_1+1} \cdots x_k^{n_k+1}
\]

\[
\sum_{n_1, \ldots, n_k \geq 1} \left[ \min(n_1, \ldots, n_k) - \min(n_1 - 1, \ldots, n_k - 1) \right] x_1^{n_1} \cdots x_k^{n_k}
\]

Now if we have that \( \min(n_1, \ldots, n_k) = a_i \) for some \( i \), then we must have that \( a_i - 1 \leq a_j - 1 \) for all \( j \). Thus \( \min(n_1 - 1, \ldots, n_k - 1) = a_i - 1 \). From this we have that \( \min(n_1, \ldots, n_k) - \min(n_1 - 1, \ldots, n_k - 1) = 1 \).

Therefore we have that

\[
(1 - x_1x_2 \cdots x_k)A(x_1, \ldots, x_k) = \sum_{n_1, \ldots, n_k \geq 1} x_1^{n_1} \cdots x_k^{n_k}
\]

\[
\left( \sum_{n \geq 1} x^n \right) \left( \sum_{n \geq 1} x^n \right) \cdots \left( \sum_{n \geq 1} x^n \right)
\]

\[
\frac{x_1}{1-x_1} \frac{x_2}{1-x_2} \cdots \frac{x_k}{1-x_k}
\]

Hence we have that

\[
A(x_1, \ldots, x_k) = \frac{x_1x_2 \cdots x_k}{(1-x_1)(1-x_2) \cdots (1-x_k)}
\]

4. (Compositions of compositions) Let \( n \) be a fixed positive integer. Find the number of ways of choosing a composition \( \alpha \) of \( n \), and then choosing a composition of \( \alpha \).

We begin by first breaking up \( n = 1 + 1 + \cdots + 1 \). Now for any composition of \( n \) we are free to remove any of the \( n-1 \)’s and combine the 1’s adjacent to each other. If we don’t remove any plus signs than we have a composition with \( n \) entries (all 1’s). Now for every plus sign that we remove we decrease the number of entries by one. Thus if we remove \( j \) plus signs we end up with a composition with \( n-j \) entries. For example let \( n = 7 \) and \( j = 3 \) then we must get \( 4 \)-compositions, some being

\[
4 + 1 + 1 + 1 \text{ or } 2 + 3 + 1 + 1 \text{ or } 2 + 2 + 2 + 1
\]