there are \( \binom{15}{1} = 15 \) ways to place this pair because there are 15 places Tolima can be (they cannot be second to last because Nacional is last) and one way to place Santa Fe in relation to Tolima.

Now we place Cali, Junior, and Huila. There are 14 open positions at this point, and so \( \binom{14}{3} \) ways to choose these teams’ three spots. Because Cali is higher than Junior which is higher than Huila, there is only one way to order these three teams once their spots are chosen.

Finally, there are 11 teams left and 11 positions open in the list. Because we have no conditions on how they are ordered, there are \( 11! \) ways to order them within the 11 positions.

Therefore, there are \( 15 \binom{14}{3} 11! \) different table of positions for the soccer league.

2a. Worked with Emily

Let \( w_n \) be the number of words of length \( n \) in the alphabet \( \{A, B, C\} \) which do not have two consecutive consonants.

If we focus on what happens at the beginning of a word of length \( n \), it can begin with an \( A \), and then there is any word of length \( n - 1 \) following it. There are no restrictions of what can follow the \( A \) because it’s a vowel. The word can also begin with a \( B \), but then it is mandatory that the next letter is an \( A \), since we cannot have two consecutive consonants. So we have \( BA \) followed by a word of length \( n - 2 \), with no restrictions of what follows the \( A \). Similarly, the word can begin with \( CA \), and be followed by any word of length \( n - 2 \).

Therefore, \( w_n \), the total number of words of length \( n \), can be given recursively by

\[
w_n = w_{n-1} + 2w_{n-2}, \quad (n \geq 2).
\]

2b. Worked with Emily

The generating function for \( w_n \) is

\[
W(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \ldots
\]

Using our recursion from part (a), we have

\[
W(x) = 1 + 3x + w_1 x^2 + w_2 x^3 + \ldots
\]

\[
+ 2w_0 x^2 + 2w_1 x^3 + \ldots
\]

The first line can be rewritten as \( 1 + 2x + xW(x) \) and the second line can be rewritten as \( 2x^2W(x) \). So,

\[
W(x) = 1 + 2x + xW(x) + 2x^2W(x)
\]

\[
\implies W(x) = \frac{1 + 2x}{1 - x - 2x^2}.
\]

2c. Worked with Emily

Using partial fractions, we want to rewrite \( W(x) \) as a geometric series, so that we will be able to explicitly see the formula for \( w_n \) as the coefficient of \( x^n \) in the series.

\[
W(x) = \frac{1 + 2x}{1 - x - 2x^2} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}
\]
for some $A, B, \alpha, \beta$. From the denominator, we get the system of equations $\alpha + \beta = 1$ and $-\alpha \beta = 2$. Using substitution, we get that $\alpha = 2$ and $\beta = -1$. We can rewrite the numerator as $(A + B) - (A\beta + B\alpha)x$, and so we get the system of equations $A + B = 1$ and $A\beta + B\alpha = -2$. Again, through substitution and using our results for $\alpha, \beta$, we get $A = \frac{4}{3}$ and $B = -\frac{1}{3}$. So,

$$W(x) = \frac{1 + 2x}{1 - x - 2x^2} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

$$\implies W(x) = A \sum_{n \geq 0} \alpha^n x^n + B \sum_{n \geq 0} \beta^n x^n = \sum_{n \geq 0} (A\alpha^n + B\beta^n) x^n$$

$$\implies w_n = A\alpha^n + B\beta^n$$

$$\implies w_n = \frac{4}{3} 2^n - \frac{1}{3} (-1)^n$$

$$\implies w_n = \frac{1}{3} (2^{n+2} + (-1)^{n+1})$$

3. Let $k$ be a fixed positive integer and consider the following identity we wish to prove:

$$\sum_{n_1, \ldots, n_k \geq 0} \min(n_1, \ldots, n_k) x_1^{n_1} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1 - x_1) \cdots (1 - x_2)(1 - x_1 x_2 \cdots x_k)}$$

We can rewrite the expression on the right:

$$\frac{x_1 x_2 \cdots x_k}{(1 - x_1) \cdots (1 - x_2)(1 - x_1 x_2 \cdots x_k)}$$

$$=(x_1 x_2 \cdots x_k) \left( \frac{1}{1 - x_1} \right) \cdots \left( \frac{1}{1 - x_k} \right) \left( \frac{1}{1 - x_1 x_2 \cdots x_k} \right)$$

$$=(x_1 x_2 \cdots x_k) \left( \sum_{n \geq 0} x_1^n \right) \cdots \left( \sum_{n \geq 0} x_k^n \right) \left( \sum_{n \geq 0} (x_1 x_2 \cdots x_k)^n \right)$$

$$=(x_1 x_2 \cdots x_k)(1 + x_1 + x_1^2 + \ldots) \cdots (1 + x_1 x_2 \cdots x_k + x_1^2 x_2^2 \cdots x_k^2 + \ldots)$$

$$=(1 + x_1 + x_2^2 + \ldots)(1 + x_2 + x_2^2 + \ldots) \cdots (x_1 x_2 \cdots x_k + x_1^2 x_2^2 \cdots x_k^2 + x_1^3 x_2^3 \cdots x_k^3 + \ldots)$$

For referencing, let’s call the first series of $x_1$’s $A_1$, the second series with $x_2$’s $A_2$, etc, and the series with $x_1 x_2 \cdots x_k$’s $A_{all}$.

Now, for the left-hand side of $\ast$, let $n_i = m$ for some $1 \leq i \leq k$ and some $m \in \mathbb{Z}$. Also, let $n_j \geq m$ for $j \neq i$, meaning $m = \min(n_1, \ldots, n_k)$. So, we want to count the number of ways get a monomial of the form $x_1^{n_1} \cdots x_i^{n_i=m} \cdots x_k^{n_k}$, for $1 \leq i \leq k$, when we multiply out the right-hand side of $\ast$. 
