in \( \mathbb{C}[x,y] \) (or \( \mathbb{R}[x,y] \)).

a) Determine whether \( x^{6} - x^{5}y \) is in \( \langle x^{3} - y, x^{2}y - y^{2}, xy^{2} - y^{3}, y^{4} - y^{2} \rangle \).

b) Check if \( \{ x^{3} - y, x^{2}y - y^{2}, xy^{2} - y^{3}, y^{4} - y^{2} \} \) is a G.B. w.r.t. \( \mathbb{C} \). or build it:

\[
\begin{align*}
S(f_{1}, f_{2}) &= yf_{1} - xf_{2} = -y^{2} + xy^{2} \equiv f_{3}, y^{2} - y^{2} = 0 \\
S(f_{1}, f_{3}) &= y^{2}f_{1} - x^{2}f_{3} = x^{2}y^{2} - y^{3} \equiv f_{2}, y^{3} - y^{3} = 0 \\
S(f_{1}, f_{4}) &= y^{3}f_{1} - x^{3}f_{4} = x^{3}y^{2} - y^{4} \equiv f_{1}, y^{3} - y^{3} = 0 \\
S(f_{2}, f_{3}) &= yf_{2} - xf_{3} = xy^{2} - y^{3} \equiv f_{1}, y^{2} - y^{3} \equiv f_{4}, 0 \\
S(f_{2}, f_{4}) &= y^{2}f_{2} - x^{2}f_{4} = x^{2}y^{2} - y^{4} \equiv f_{2}, y^{3} - y^{4} \equiv f_{1}, y^{3} - y^{3} = 0 \\
S(f_{3}, f_{4}) &= yf_{3} - xf_{4} = xy^{2} - y^{3} \equiv f_{3}, y^{2} - y^{3} \equiv f_{4}, 0 \\
\end{align*}
\]

\( \Rightarrow \{ f_{1}, f_{2}, f_{3}, f_{4} \} \) is a G.B. w.r.t. \( \mathbb{C} \).

2) Determine whether \( x^{6} - x^{5}y \in I \):

\[
\begin{align*}
g_{1} &= g_{0} - x^{3}f_{1} = x^{3}y - x^{5}y \\
g_{2} &= g_{1} + x^{2}f_{1} = x^{3}y - x^{2}y^{2} \\
g_{3} &= g_{2} - yf_{1} = y^{2} - x^{2}y^{2} \\
g_{4} &= g_{3} + yf_{2} = y^{2} - y^{3} \\
g_{5} &= g_{4} + f_{4} = 0 \\
\end{align*}
\]

\( \Rightarrow x^{6} - x^{5}y \in I \).
b) Determine whether \( \langle x^3 - y^2, y^2 + y \rangle = \langle x^2 + x^3, y^3 + y \rangle \).

1. Determine whether \( \{f_1, f_2 \} \) and \( \{g_1, g_2 \} \) are a Gröbner basis:

   \[
   S(f_1, f_2) = y^2 f_1 - x^2 f_2 = -x^3 y - y^2 z^2 \equiv f_1 - y^2 z^2 - y^2 z^2 \equiv f_2 - y^2 z + y^2 z = 0
   \]

   \[
   S(g_1, g_2) = g_1 - 2g_2 = x^3 - 2y \equiv g_2 - y - 2y \quad \Rightarrow \quad \text{take } g_3 = 2y + y
   \]

   \[
   S(g_1, g_3) = yg_1 - x^3 g_3 = x^3 y - x^3 y = 0
   \]

   \[
   S(g_2, g_3) = 2yg_2 - xg_3 = 2y^2 - x^3 y \equiv g_2 - y^2 z + y^2 z \equiv g_3 - y^2 z + y^2 z = 0
   \]

   \( \Rightarrow \) \( \{f_1, f_2 \} \) is a G.b for \( \langle f_1, f_2 \rangle \) and \( \{g_1, g_2, g_3 \} \) is a G.b for \( \langle g_1, g_2 \rangle \).

2. Construct/check if the G.b are reduced: all \( \langle f_1 \rangle \) are monic.

   - Since \( \text{in}(f_2) = y^2 \) divides a monomial in \( f_1 \), we replace \( f_1 \) by its remainder dividing by \( f_2 \) \( \Rightarrow \) \( r = f_1 + f_2 = x^3 + y^2 \).

   \( \Rightarrow \) \( \{x^3 + y^2, y^2 + y^3 \} \) is the reduced G.b for \( \langle f_1, f_2 \rangle \).

   - Since \( \text{in}(g_2) \) divides monomials in \( g_1 \), we replace \( g_1 \) by the remainder dividing by \( \{g_2, g_3 \} \):

     \[
     g_1 = 2g_2 + g_2 - g_3 + 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad \{g_2, g_3 \} \text{ is the reduced G.b for } \langle g_1, g_2 \rangle.
     \]

3. Compare the reduced G.b: Since the reduced G.b for \( \langle f_1, f_2 \rangle \) and \( \langle g_1, g_2 \rangle \) w.r.t lex are equal then the two ideals are equal.
6. Solve the system of equations \( x^2 - y^2 = 3, \ y^2 - x^2 = 9, \ 2^2 - xy = 5 \).

7. Compute a G.B. for \( \langle x^2 - y^2 - 3, \ x^2 - y^2 + 4, \ xy - 2^2 + 5 \rangle \).

\[
S(f_1, f_2) = 2f_1 - x f_2 = -y^2 - 3x + xy^2 - 4x \equiv f_3 - y^2 - 3x + \frac{y^2}{4} - 5y - 4x
\]

\[
f_4 = 4x + 5y + 3z
\]

\[
S(f_1, f_3) = y f_1 - x f_3 = \frac{x^2}{2} - 5x - y^2 - 3y \equiv f_2 \ y^2 - 4z - 5x - y^2 - 3y
\]

\[
\equiv f_5 - 3y - 4z + \frac{25}{4}y + \frac{15}{4} z = \frac{13}{4} y - \frac{1}{4} z
\]

\[
\Rightarrow f_5 = 13y - 2
\]

\[
S(f_2, f_3) = y f_2 - 2 f_3 = -y^3 + 4y + 2^3 - 5z \equiv f_5 - \frac{1}{13} 2^3 + 4y + 2^3 - 5z
\]

\[
\equiv f_5 \ \frac{13^2 - 1}{13^2} 2^3 + \frac{4}{13} 2 - 5z = \frac{13^2 - 1}{13^2} 2^3 + \frac{4-5.12}{13} z \quad \Rightarrow f_6 = \frac{(13^2 - 1) 2^3 + 13^2 (4-5.12)}{2}
\]

\[
S(f_1, f_4) = f_1 - \frac{1}{4} x f_4 = -y^2 - 3 - \frac{5}{4} x y - \frac{3}{4} x z \equiv f_3 - y^2 - 3 - \frac{5}{4} z + \frac{25}{4} - \frac{3}{4} z
\]

\[
\equiv f_2 - y^2 + \frac{13}{4} - \frac{5}{4} z - \frac{3}{4} y^2 + 3 \equiv f_5 - y^2 + \frac{25}{4} - \frac{5}{4} z - \frac{3}{4} z
\]

\[
\equiv f_2 - \frac{13}{4} + \frac{25}{4} - \frac{5}{4} z - \frac{3}{4} z = \frac{-225}{169} + \frac{25}{4} \Rightarrow f_6 = \frac{362^2}{169}
\]

\[
(f_6 \equiv 0 \pmod{362^2 - 169})
\]

\[
S(f_2, f_4) = f_2 - \frac{1}{4} f_2 = -y^2 + 4 - \frac{5}{4} y^2 - \frac{3}{4} z \equiv f_5 - \frac{1}{13} 2^2 + 4 - \frac{5}{4} y^2 - \frac{3}{4} z
\]

\[
\equiv f_5 - \frac{1}{13} 2^2 + 4 - \frac{5}{4} \cdot \frac{13}{4} \ 2^2 - \frac{3}{4} z = f_6 - \frac{1}{4} 2^2 - 4 - \frac{5}{4} \cdot \frac{13}{4} \ 2^2 - \frac{13^2}{4^2} \ 3 = 0
\]
\[ \text{S}(f_3, f_4) = f_3 - \frac{1}{4} y f_4 = x y - z^2 + 5 - x y - \frac{5}{4} y^2 - \frac{3}{4} y^2 \]

\[ \equiv f_5 - 2^2 + 5 - \frac{5}{9.13^2} z^2 - \frac{3}{4} y^2 \equiv f_5 - 2^2 + 5 - \frac{5}{9.13^2} z^2 - \frac{3}{4.13} z^2 \]

\[ \equiv f_6 - \frac{13^2}{4.9} + 5 - \frac{5}{4.9^2} q - \frac{4.13}{4.9^2} q = 0 \]

\[ \text{S}(f_1, f_5) = y f_1 - \frac{1}{13} x^2 f_5 = -y^2 z - 3 y + \frac{1}{13} x^2 z \equiv f_1 - y^2 z - 3 y - \frac{1}{13} y^2 + \frac{3}{13} z \]

\[ \equiv f_5 - \frac{1}{13} z^2 - 3 y + \frac{1}{13} y^2 + \frac{3}{13} z \equiv f_6 - \frac{1}{13} z^2 - 3 y + \frac{13}{4.9} q + \frac{3}{13} z \]

\[ \equiv f_5 - \frac{1}{13} z^2 - \frac{3}{13} z + \frac{1}{4.9} z + \frac{3}{18} z \equiv f_6 - \frac{4}{4.9} q + \frac{1}{4.9} z = 0 \]

\[ \text{S}(f_2, f_5) = y f_2 - \frac{x^2 f_5}{13} = -y^3 + 4 y + \frac{x^2}{13} \equiv f_2 - y^3 + 4 y + \frac{4 y^2}{13} - \frac{4}{13} z \]

\[ \equiv f_5 - \frac{1}{13} y^2 + 4 y + \frac{4.13}{13} z \equiv f_6 - \frac{1}{13} y^2 + 4 y + \frac{4}{13} z \]

\[ \equiv f_5 \frac{4}{13} z - \frac{4}{13} z = 0 \]

\[ \text{S}(f_3, f_5) = f_3 - \frac{1}{13} x f_5 = -z^2 + 5 + \frac{1}{13} x z \equiv f_2 - z^2 + 5 + \frac{1}{13} y^2 - \frac{4}{13} \equiv f_5 - z^2 + 5 + \frac{1}{13} z - \frac{4}{13} \]

\[ \equiv f_6 - \frac{13^2}{4.9} + 5 + \frac{1}{13} q - \frac{4}{13} = 0 \]

\[ \text{S}(f_4, f_5) = \frac{1}{4} y f_4 - \frac{1}{13} x f_5 = -\frac{5}{4} y^2 + \frac{3}{9} y^2 + \frac{1}{13} x z \equiv f_2 - \frac{5}{4} y^2 + \frac{3}{9} y^2 + \frac{1}{13} y^2 - \frac{4}{13} \]

\[ \equiv f_5 \frac{5}{4.13^2} z^2 + \frac{3}{13} y^2 + \frac{1}{13} z^2 - \frac{4}{13} \equiv f_6 \frac{5}{4.13^2} z^2 + \frac{3}{4.13} z^2 + \frac{1}{13} z^2 - \frac{4}{13} \]

\[ \equiv f_6 \frac{5}{4.9^2} + \frac{13.3}{9.9^2} + \frac{1}{13.9.9} - \frac{4}{13} = 0 \]
\[
S(f_1, f_6) = 2\ f_1 - \frac{1}{9} x^2 f_6 = -y^2 z - 3z^2 + \frac{13}{9} z + \frac{13}{9} y^2 + \frac{13}{9} z
\]

\[
\equiv f_6 - \frac{13}{9} y^2 z - 3z^2 + \frac{13}{9} y^2 + \frac{13}{9} z \equiv f_6 - \frac{13}{9} z + \frac{13}{9} z = 0
\]

\[
S(f_2, f_6) = 2\ f_2 - \frac{1}{9} x f_6 = -y^2 z + 4z + \frac{13}{9} z \equiv f_4 - y^2 z + 4z - \frac{13}{9} y - \frac{13}{9} z
\]

\[
\equiv f_4 - \frac{13}{9} z + 4z - \frac{13}{9} y - \frac{13}{9} z \equiv f_4 - \frac{13}{9} z + 4z - \frac{13}{9} y - \frac{13}{9} z = 0
\]

\[
S(f_3, f_6) = 2\ f_3 - \frac{1}{9} x y f_6 = -2^2 + 5z^2 + \frac{13}{9} y \equiv f_5 - 2^2 + 5z^2 + \frac{13}{9} y
\]

\[
\equiv f_5 - 2^2 + 5z^2 + \frac{13}{9} y \equiv f_5 - 2^2 + 5z^2 + \frac{13}{9} y
\]

\[
S(f_4, f_6) = \frac{1}{4} 2^2 f_4 = \frac{1}{4} x f_6 = \frac{5}{9} y^2 + \frac{3}{9} z^2 + \frac{13}{9} z
\]

\[
\equiv f_6 - \frac{5}{9} y^2 + \frac{3}{9} z^2 + \frac{13}{9} z \equiv f_6 - \frac{5}{9} y^2 + \frac{3}{9} z^2 + \frac{13}{9} z
\]

\[
S(f_5, f_6) = \frac{1}{13} 2^2 f_5 - \frac{1}{9} y f_6 = -\frac{1}{13} 2^3 + \frac{12}{9} y \equiv f_5 - \frac{1}{13} 2^3 + \frac{12}{9} y \equiv f_5 - \frac{1}{13} 2^3 + \frac{12}{9} y
\]

\[
\equiv f_5 - \frac{1}{13} 2^3 + \frac{12}{9} y \equiv f_5 - \frac{1}{13} 2^3 + \frac{12}{9} y
\]

\[
- \frac{1}{13} 2^3 + \frac{12}{9} y + \frac{13}{9} z = 0
\]

\[
- \frac{1}{13} 2^3 + \frac{12}{9} y + \frac{13}{9} z = 0
\]

\[
\equiv \{ f_1, f_2, f_3, f_4, f_5, f_6 \} \text{ is a G.B. for } \langle f_1, f_2, f_3 \rangle \text{ actually we}
\]

\[
\text{will consider the Gröbner Basis } G = \{ f_1, f_5, f_6 \} \text{ for this ideal (since}
\]

\[
\text{the initial terms of these monomials divide those of } f_1, f_2).
\]
2. Solve the system of equations:
   
a) Given \( F(z) = 3z^2 - 19 \), the solutions to this equation are:
   \[ z = \pm \frac{13}{6} \]

b) Given \( F(x, y, z) = \{ 13y - z, 36z^2 - 2 \} \), to solve the system:
   \[ 13y - z = 0 \]  
   \[ 36z^2 - 2 = 0 \]
   We substitute \( z = \pm \frac{13}{6} \) and solve the system.
   Then, the points \( \left( \frac{1}{6}, \frac{13}{6} \right) \) and \( \left( -\frac{1}{6}, -\frac{13}{6} \right) \) are the solutions to the system.

c) Given \( F(x, y, z) = \{ 9x + 5y + 2z, 13y - z, 36z^2 - 19 \} \), and the solutions for the system given by these equations can be found by substituting the previous points into the first equation and finding \( x \).

   \[ 4x + \frac{5}{6} + \frac{39}{6} = 0 \]  
   \[ 4x = -\frac{11}{6} \]

   \[ 4x - \frac{5}{6} - \frac{39}{6} = 0 \]  
   \[ x = \frac{11}{6} \]

   Therefore, the solutions to the system are the points
   \[ P_1 = \left( -\frac{11}{6}, \frac{5}{6}, \frac{13}{6} \right) \]  
   \[ P_2 = \left( \frac{11}{6}, -\frac{1}{6}, -\frac{13}{6} \right) \]

Note: Here we used the fact that \( V(\langle f_1, \ldots, f_k \rangle) = V(\langle f_{j_1}, \ldots, f_{j_k} \rangle) \) because we found:

\[ V(\langle f_1, f_5, f_6 \rangle) = V(\langle f_1, f_3, f_3 \rangle) = V(\langle f_1, f_2, f_3 \rangle) = \{ P_1, P_2 \} \].
d) Compute \(<x^3 y - x y^2 + 1, x^2 y^2 - y^3 - 1> \cap <x^2 - y^2, x^3 + y^3>\).

\(<t, 1-t> \supset <t (x^3 y - x y^2 + 1), t (x^2 y^2 - y^3 - 1), (1-t) (x^2 - y^2), (1-t) (x^3 + y^3)>\)

Let's compute a G. b. for this ideal w.r.t. \( t > x > y \).

\(S(f_1, f_2) = y f_1 - x f_2 = -t x y^3 + t y + t x y^3 + t x \Rightarrow f_5 = t x + t y\)

\(S(f_1, f_3) = f_1 + x y f_3 = -t x y^2 + t + x^3 y - x y^3 + t x y^3 \equiv f_5 - t x y^2 + t + x^3 y - x y^3 - t y^4 \)

\(\equiv f_5 (t y^3 + t + x^3 y - x y^3 - t y^4) \Rightarrow f_6 = t y^3 - t + y y^3\)

\(S(f_1, f_4) = f_1 + y f_4 = -t x y^2 + t + x^3 y + y^4 - t y^4 \equiv f_5 t y^3 + t + x^3 y + y^4 - t y^4 \)

\(\equiv f_6 (t y^3 + t + x^3 y + y^4 - t y^4) \Rightarrow f_7 = x y^3 + y^4\)

\(S(f_2, f_3) = f_2 + y^2 f_3 = -t y^3 - t + x^2 y^2 - y^4 + t y^4 \equiv f_6 t x^2 - x + x^2 y^2 - y^4 + t y^3 + t x^3 y - x y^3 \)

\(= x^2 y^3 - x y^3 + y^4 \Rightarrow f_8 = x^2 y^3 + x y^3 - y^4\)

\(S(f_2, f_4) = x f_2 + y^2 f_4 = -t x y^3 - t x + x^2 y^2 + y^4 - t y^5 \equiv f_5 t x - t x y - t + t x^3 y + y^4 \)

\(\equiv f_6 t y^4 + t y - t y^4 + x y^2 + t y^5 \equiv f_6 t x y + t y^5 = 0\)

\(S(f_3, f_4) = x f_3 - f_4 = x^3 x y^2 + t x y^2 - x^3 y^3 + t y^3 \equiv f_6 - t x y^2 - y^3 + t y^3 \)

\(\Rightarrow f_8 = x y^2 + y^3\) (This \( f_8 = y(x y^2 + y^3) \))

\(S(f_5, f_1) = f_1 - x^2 y f_5 = -t x y^2 + t - t x y^2 \equiv f_2 - t x y^2 + t - t y^3 + t \equiv 0\)

\(S(f_5, f_5) = f_1 - x^2 y f_5 = -t x y^2 + t - t x y^2 \equiv f_2 - t x y^2 + t - t y^3 + t \equiv 0\)
\( S(f_1, f_5) = f_2 - xy^2 f_5 = -ty^3 - t - \frac{txy^3}{f_5} - ty^3 + by^4 = f_6 - ty^3 + ty^3 + ty^3 - xy^3 \\
\equiv f_8 - x^2 y^2 + xy^3 + y^4 - xy^3 \equiv f_2 + xy^3 + y^4 = f_7 + 0 \\
S(f_5, f_5) = f_3 + xy f_5 = x^2 - y^2 + ty^2 + txy \equiv f_5 x^2 - y^2 + ty^2 - by \\
\equiv f_3 = (x - y)(x^2 - y^2) \\
S(f_5, f_6) = f_4 + xy f_6 = x^3 + y^3 - tx^3 + tx^3 + t x^3 + x^6 y - x^4 y^3 \\
\equiv f_5 x^3 + y^3 - tx^3 - x^3 y^3 = f_5 x^3 + y^3 \equiv f_4 + 0 \\
S(f_1, f_6) = y^2 f_2 - x^3 f_6 = -ty^3 - ty^3 + t x^3 y + t x^3 y + x^3 y - x^3 y^3 \\
\equiv f_1 - ty^3 + ty^3 + t x^3 y + t x^3 y - x^3 y^3 = f_5 t x^3 - ty^3 + ty^6 + x^3 y - x^3 y^3 \\
\equiv f_1 - ty^3 + ty^6 + x^3 y - x^3 y^3 = f_3 + 0 \\
S(f_3, f_6) = ty^3 f_3 - x^2 f_6 = -tx^3 y^3 + t x^2 y^3 + t x^2 y + x^3 y - x^3 y^3 \\
\equiv f_6 - bx^3 y^3 + ty^3 - x^3 y^3 + t x^3 y^3 + x^3 y - x^3 y^3 = x^3 y - 2x^3 y^3 + x y^5 \\
\equiv f_3 - x^3 y + xy^5 = f_3 + 0 \\
S(f_3, f_4) = y^2 f_4 + x^3 f_6 = x^3 y^4 + y^7 - by^7 - tx^3 y^3 - tx^3 - x^6 y + x^3 y^3 = f_3 x^3 y^4 + y^7 - tx^3 y^3 - tx^3 - x^6 y + x^3 y^3 \\
\equiv f_3 x^3 y^4 + y^7 - by^7 - tx^3 - tx^3 - x^6 y + x^3 y^3 \\
\equiv f_6 x^3 y^4 + y^7 - by^7 - tx^3 = x^3 y^4 + xy^6 + by^7 + t y^7 - x^4 y + x^4 y^3 \\
= -x^6 y + x^4 y^3 + xy^6 + y^7 \equiv f_3 - x^4 y^3 + x^4 y^3 + xy^6 + y^7 = f_3 + 0 \)
$S(f_5, f_6) = y f_5 - x f_6 = -x y^3 + t x^2 y + t x + x^4 y - x^2 y^3 \equiv f_7 ty^4 + t x^2 y - x^2 y^3$

$\equiv f_6 ty^4 - t y^4 + t x^2 y - t y^4 - x^2 y^3 \equiv f_6 ty^4 + t x^2 y - x^2 y^3$

$\equiv f_5 x^2 y^2 - x y^4 + x^4 y - x^2 y^3 \equiv f_3 x y^2 - x y^4 \equiv f_3 0$

$S(f_7, f_8) = y f_7 - t x y^3 - t x y^3 \equiv f_3 - t x y^3 + t y^4 \equiv f_6 ty^4 + t y^4$

$\equiv f_6 ty^4 + t y^4 - x^2 y^2 + x y^4 \equiv f_3 0$

$S(f_4, f_5) = x^2 f_4 - x f_5 = -t x y^3 + t y^4 - x^2 y^3 \equiv f_3 - t x y^3 + t y^4 \equiv f_5 ty^4 + t y^4 - x^2 y^3$

$\equiv f_6 ty^4 + t y^4 - x^2 y^4 + x y^4 \equiv f_3 x y^2 + x y^4 \equiv f_3 0$

$S(f_3, f_6) = y^2 f_3 - x f_6 = -y^4 - x y^3 \equiv f_3 0$

$S(f_3, f_7) = y^2 f_4 + t x^2 f_7 = x^3 y^2 + x y^4 - t x y^3 + t x^2 y^3 \equiv f_3 x y^2 + x y^4 \equiv f_3 x y^2 + x y^4 \equiv f_3 0$

$S(f_3, f_8) = y f_3 - t f_8 = t x y^3 + t x y^3 + t x y^4 \equiv f_3 - t x y^3 + t x y^4 \equiv f_3 0$

$S(f_5, f_8) = x f_5 - t f_8 = -t x y^3 - t x y^3 + t x y^4 + t y^4 \equiv f_3 - t x y^3 + t x y^4 \equiv f_3 0$

$S(f_4, f_8) = y f_4 + t f_8 = x y^2 + x y^4 - t x y^3 + t x y^4 \equiv f_3 x y^2 + t x y^3$

$\equiv f_3 ty^4 + t y^4 \equiv 0$

$S(f_6, f_8) = x f_6 - t f_8 = -t x y^3 - t x y^3 + t x y^4 + t y^4 \equiv f_3 - t x y^3 + t x y^4 \equiv 0$

$S(f_7, f_8) = y f_7 - t f_8 = -t x y^3 - t x y^3 + t x y^4 + t y^4 \equiv f_3 - t x y^3 + t x y^4 \equiv 0$

$S(f_8, f_8) = y f_8 + t f_8 = x y^2 + x y^4 - t x y^3 + t x y^4 \equiv f_3 x y^2 + x y^4 \equiv f_3 0$

$S(f_3, f_8) = x^2 f_3 - t f_8 = t x^2 y - t x y^3 + t x y^3 + t x y^4 \equiv f_3 0$
\((a, f_3) = x f_3 - y^3 f_3 = -txy^3 - tx^3 - x^4 y + x^5 y - txy^5 + txy^6 + ty^7\)

\[ x f_3 = -txy^3 - tx^3 - x^4 y + x^5 y + x^6 y + by^6 \]
\[ \equiv f_3 \quad \text{by } \frac{f_3}{y^3} = \frac{tx^3}{y^3} - x^3 y + 3y^3 - x^3 y + xy^6 \]
\[ \equiv f_3 \quad \frac{x^3 y^3 + y^6}{f_3} = 0 \]

\(S(f_3, f_2) = x f_3 - y f_3 = x^2 - x y + y^4 + y^5 \equiv f_3 \)

\(\Rightarrow \{ f_1, \ldots, f_6 \} \text{ is a Gröbner basis for the desired ideal} \)

And thus \(\text{In} \mathcal{I} = \langle \text{GNI } f_1, f_2, f_3 \rangle = \langle x^2 - y^3, x y^2 + y^3, x^3 y + x y^3 - x y^2 - y^4 \rangle \).

(6) Compute the syzygies between the polynomials \(x^2 y^3\) and \(x y + y^2\).

\(S(f_3, f_2) = y f_3 - x^2 f_3 = 0 \equiv \begin{bmatrix} y^2 \\ -x \\ -y \end{bmatrix} \) is a syzygy

\(S(f_1, f_3) = f_1 + x f_3 = -xy^2 = -z f_3 + y^2 \quad \Rightarrow \begin{bmatrix} f_3 = y^2 \\ f_4 = y f_1 + (z-x) f_3 \end{bmatrix} \)

\(S(f_1, f_3) = x f_2 - y f_3 = -y^2 z = -2 f_2 \quad \Rightarrow \begin{bmatrix} 0 \\ x + z \\ y \end{bmatrix} \) is a syzygy

\(S(f_4, f_1) = y z^2 f_1 - x^2 f_4 = 0 \)

\(S(f_2, f_4) = x^2 f_2 - y f_4 = 0 \)

\(S(f_3, f_4) = 2 f_3 - x f_4 = y^2 = 2 f_4 \quad \Rightarrow \begin{bmatrix} 2 f_3 \quad (x+z) f_4 = 0 \end{bmatrix} \)

Since we introduced \(f_4\) we need to substitute \(f_4\) in each of the relations found.

From \(S(f_1, f_4)\) we get \(y z^2 f_1 - x^2 f_1 + x^2 (z-x) f_3 = 0\) so we get the syzygy \(\begin{bmatrix} y^2 - x^2 y \\ 0 \\ x^3 - x^2 z \end{bmatrix} \).
From $S(f_2, f_4)$ we get \[ z^2 f_2 - y^2 f_1 + (xy - yz) f_3 = 0 \] and thus the syzygy
\[
\begin{bmatrix}
-y^2 \\
z^2 \\
x y - yz
\end{bmatrix}.
\]

From $S(f_3, f_4)$ we get \[ z^2 f_3 - y(x^2 + z) f_1 + (x+z)(x-z) f_3 = 0 \] and thus the syzygy
\[
\begin{bmatrix}
-xy - yz \\
0 \\
x^2
\end{bmatrix}.
\]

Therefore, the syzygies among the given polynomials are generated by
\[
\left\{ \begin{bmatrix} -y^2 & \cdot & \cdot \\ -xy & \cdot & \cdot \\ x^2 & \cdot & \cdot \end{bmatrix} \right\}.
\]