By the last item and transitivity, it is enough to prove that $0 < \beta e_i$ for every $1 \leq i \leq n$: the canonical vectors are bounded below by 0. Suppose $v_t$ is the first vector in $\beta$ with nonzero $i$-th coordinate, which exists because $\dim(\text{span}(\beta)) = n$, then $(v_t)_i > 0$ and

$$v_j \cdot e_i = v_j \cdot 0 = 0 \text{ if } j < t \text{ and }$$

$$v_t \cdot e_i = (v_t)_i > 0 = v_t \cdot 0$$

so $0 < \beta e_i$ for all $i$, as we wanted.

d. Suppose we have a monomial ordering $<_{\text{mon}}$ and let $C$ be the set of vectors $\nu \in \mathbb{Z}^n$ such that $\nu = a - b$ for monomials $x^b <_{\text{mon}} x^a$.

Result 1. If $v_1, v_2 \in C$ and $p, q \in \mathbb{Q}_+$ then $pv_1 + qv_2 \in C$ whenever $pv_1 + qv_2 \in \mathbb{Z}^n$.

Proof. Consider four monomials $x^b <_{\text{mon}} x^a$ and $x^d <_{\text{mon}} x^c$ in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose we have $p, q \in \mathbb{Q}_+$ such that $\nu = p(a - b) + q(c - d) \in \mathbb{Z}^n$. It is convenient to explicitly write $p$ and $q$ as fractions:

$$p = \frac{p_N}{p_D}, \quad q = \frac{q_N}{q_D} \quad \text{with } p_D, q_D, p_N, q_N \in \mathbb{Z}_+$$

We know $(x^b)^{pNqD} <_{\text{mon}} (x^a)^{pNqD}$ and $(x^d)^{pDqN} <_{\text{mon}} (x^c)^{pDqN}$. Defining

$$f = pNqDa + pDqNa$$

$$g = pNqDb + pDqNd$$

we have $x^g <_{\text{mon}} x^f$ so $f - g \in C$. It is easy to check that $-\mu \notin C$ whenever $\mu \in C$. Now, choose $e \in \mathbb{Z}^n_{\geq 0}$ so that $e + \nu \in \mathbb{Z}^n_{\geq 0}$, then $(x^e)^{pDqD} = x^{pDqDe}$ and $(x^{e+\nu})^{pDqD} = x^{pDqDe+f-g}$. Because $f - g$ is in $C$ and $<_{\text{mon}}$ is a total ordering on monomials, we have $x^{pDqDe} <_{\text{mon}} x^{pDqDe+f-g}$ so $(x^e)^{pDqD} <_{\text{mon}} (x^{e+\nu})^{pDqD}$. But then it must be true that $x^e <_{\text{mon}} x^{e+\nu}$ so $\nu \in C$. \qed

Suppose we are given a finite number of vectors $\nu_1, \nu_2, \ldots, \nu_m \in C$ and consider the convex combination

$$\sum_{i=1}^{m} r_i \nu_i = 0$$
with $r_1, \ldots, r_m \in \mathbb{R}^+$. Clearly $m \geq 2$. Let $M$ be the $n \times m$ matrix with $i$-th column equal to $\nu_i$ and let $r = (r_1, \ldots, r_m)$. The null space of the linear map $L_M$ induced by $M$ is nontrivial because $L_M(r) = 0$ and has a basis $\beta$ consisting of vectors in $\mathbb{Q}^m$. This may be checked by Gaussian elimination on $M$. Say $\beta = \{u_1, \ldots, u_l\}$ and write $r = c_1u_1 + \cdots + c_lu_l$. As $r$ lies in $\mathbb{R}^m_+$ we may find rationals $c_1^*, \ldots, c_l^*$ sufficiently close to $c_1, \ldots, c_l$ (respectively) such that $q = c_1^*u_1 + \cdots + c_l^*u_l$ has positive components, i.e. $q \in \mathbb{Q}^m_+$. We can choose these rational coefficients so that the components of $q$ add up to 1. But then $L_M(q) = 0$ and we have the rational convex combination

$$\sum_{i=1}^{m} (q)_i \nu_i = 0$$

More conveniently, we have positive integers $n_1, \ldots, n_m$ such that

$$\sum_{i=1}^{m} n_i \nu_i = 0$$

Using Result 1 inductively we obtain a contradiction because clearly $0 \notin C$. Note Equation 1 holds for vectors $\nu_1, \ldots, \nu_m \in C$ iff $0$ lies in the convex hull of $C$, denoted by $\text{ch}(C)$. Thus $0 \notin \text{ch}(C)$.

One separation theorem in convex analysis shows there exists an hyperplane $H^{(1)}$ of $\mathbb{R}^n$ separating (not necessarily strictly) 0 and $\text{ch}(C)$, i.e. there exists a vector $v_1 \in \mathbb{R}^n \setminus \{0\}$ such that $v_1 \cdot x \geq 0$ for all $x \in \text{ch}(C)$. Actually $v_1$ has nonnegative components because $C$ contains the canonical basis of $\mathbb{R}^n$. Moreover, if $\mu$ is a vector in $\mathbb{Z}^n \setminus \{0\}$ such that $v_1 \cdot \mu > 0$ then $\mu$ lies in $C$ because either $\mu \in C$ or $-\mu \in C$ holds and in the later case we would have $v_1 \cdot \mu \leq 0$. We may simply define $H^{(1)} = v_1^\perp$, the orthogonal complement of $v_1$.

Now, define the set $C^{(1)} = H^{(1)} \cap C$. If $C^{(1)} = \emptyset$ then we are done and we can tell apart any element of $C$ via taking the dot product with $v_1$. Otherwise, notice that $0 \notin \text{ch}(C^{(1)})$. Extending the separation theorem to arbitrary vector subspaces of $\mathbb{R}^n$ we can find an hyperplane $H^{(2)}$ of $H^{(1)}$ separating 0 and $\text{ch}(C^{(1)})$, or equivalently, a vector $v_2 \in H^{(1)} \setminus \{0\}$ such that $v_2 \cdot x \geq 0$ for all $x \in \text{ch}(C^{(1)})$ so that $H^{(2)} = v_1^\perp \cap v_2^\perp$. Again, if for some $\mu \in \mathbb{Z}^n \setminus \{0\}$ we have $v_1 \cdot \mu = 0$ and $v_2 \cdot \mu > 0$, then $\mu$ lies in $C^{(1)} \subseteq C$. We then define $C^{(2)} = H^{(2)} \cap C^{(1)}$ and ask whether $C^{(2)} = \emptyset$. If true, we can tell apart vectors in $C$ via first taking their dot product with $v_1$ and, in case of a 0, subsequently taking the product with $v_2$, which would suffice. If not true, we continue our inductive
process until we stop. It will have to stop necessarily when we find \( n \) vectors \( v_1, v_2, \ldots, v_n \); they are pairwise orthogonal by construction, so by linear independence they form a basis of \( \mathbb{R}^n \) and we would have \( C^{(n)} = H^{(n)} \cap C^{(n-1)} = (v_1^\perp \cap v_2^\perp \cap \cdots \cap v_n^\perp) \cap C^{(n-1)} \subseteq \{0\} \cap C = \emptyset \). In any case, we could complete the orthogonal basis without affecting the weight order.

5. Notice \( \text{in}_<(I) = \langle \text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_m) \rangle \). If \( h \) is in \( \text{in}_<(I) \) then \( h = \sum_{i=1}^m f_i \text{in}_<(g_i) \) for some \( f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n] \). Writing \( f_i = \sum_{j=1}^{n_i} c_{ij} x^{a_{ij}} \) we have \( h = \sum_{i=1}^m \sum_{j=1}^{n_i} c_{ij} \langle x^{a_{ij}} \rangle \text{in}_<(g_i) \) which is a sum of monomials each of which is divisible by \( \text{in}_<(g_i) \) for some \( i \). Now suppose \( h \) can be written as a sum of monomials, each one of which is divisible by \( \text{in}_<(g_i) \). Ordering terms and factoring we then can write \( \sum_{i=1}^m \left( \sum_{j=1}^{n_i} c_{ij} x^{a_{ij}} \right) \text{in}_<(g_i) \) or \( \sum_{i=1}^m f_i \text{in}_<(g_i) \) for some \( f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n] \). Therefore, \( h \) belongs to \( \text{in}_<(I) \).