The Symmetries of a Square

The goal of this investigation is to describe all of the symmetries of a square and all of the interactions between these symmetries. Remember that a symmetry of a square is a rigid motion of the plane which leaves the outline of the square unchanged. (Note that a 90° rotation about the center point of a square is a symmetry of the square, but a 45° rotation about the center point of a square is not a symmetry of the square.)

Introduction

You will need to make a square of paper the size of the square silhouetted on this sheet of paper for your investigation. Label the corners of the cut-out square exactly as they are labeled on the silhouette (With W, X, Y, and Z). The back side of the square should also be labeled so that, for example, the “Z” is directly under the “Z” on the front side of the cut-out square.

The HOME position of the cut-out square is a position on top of the square below, with labeled corners touching labeled corners.

When you are using the cut-out and its silhouette, it is important to describe and/or act out all rotations and reflections based on the position of the silhouette. The descriptive points and lines on the diagram below will always be your frame of reference. The four lines labeled
Part 1 - DESCRIBING THE SYMMETRIES

1. Use words to describe the 8 symmetries of the square. These symmetries are written in mathematical shorthand below.

\[ R_0 \]

\[ R_{90} \]

\[ R_{180} \]

\[ R_{270} \]

\[ H \]

\[ V \]

\[ D \]

\[ D' \]

2. The rotation \( R_{90} \) can be thought of as a function from \( \{W,X,Y,Z\} \) to \( \{W,X,Y,Z\} \) and described by an input-output table. The first column of the table is interpreted as follows: Beginning with the cutout in home position, the function \( R_{90} \) sends vertex \( W \) of the cutout to position \( X \) of the silhouette. Fill in the three missing outputs, and create similar input-output tables for the other seven symmetries.

\[ \begin{align*}
R_{90} := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
R_0 := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
R_{180} := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
R_{270} := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
H := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
V := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
D := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]

\[ \begin{align*}
D' := & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right) \\
\leftrightarrow & \left( \begin{array}{c}
W \\
X \\
Y \\
Z \\
\end{array} \right)
\end{align*} \]
Part 2 - Describing interactions between the symmetries of the square.

1. We use **composition of functions** to define an operation on the set of the 8 symmetries of the square.

   NOTATION: Juxtaposition of two of the symbols for symmetries implies composition. We apply the functions **from right to left**: so the notation $DR_{90}$ means first rotate the cut-out square 90 degrees clockwise, and then reflect the square about the main diagonal of the silhouette.

   Complete the input-output table for $DR_{90}$.

   $DR_{90} := \begin{align*} 
   \text{In} \left( \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \right) \\
   \text{Out} \left( \begin{array}{c} Z \\ \end{array} \right) 
   \end{align*}$

   Compare this input-output table to the input-output tables that you created on the second page and fill in the blank, $DR_{90} =$ ________

   Repeat this exercise for $R_{90}D$.

   $R_{90}D := \begin{align*} 
   \text{In} \left( \begin{array}{c} W \\ X \\ Y \\ Z \end{array} \right) \\
   \text{Out} \left( \begin{array}{c} X \\ \end{array} \right) 
   \end{align*}$

   Fill in the blank: $R_{90}D =$ ________

2. Determine the results of all of the possible compositions of the symmetries of the square. The table that you create is called the Cayley table of a mathematical system called the group of symmetries of the square. (The term “**group**” is used here because we have a set of objects - the symmetries - and a nice binary operation - composition of functions - on which to operate.) What you are doing is something like creating a multiplication table.

   **The Cayley Table for the Group of symmetries of a square.**

<table>
<thead>
<tr>
<th>Composition</th>
<th>R0</th>
<th>R90</th>
<th>R180</th>
<th>R270</th>
<th>H</th>
<th>V</th>
<th>D</th>
<th>D'</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0</td>
<td>In</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R90</td>
<td></td>
<td>Out</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R180</td>
<td></td>
<td></td>
<td>Out</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R270</td>
<td></td>
<td></td>
<td></td>
<td>Out</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>In</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>In</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>In</td>
<td></td>
</tr>
<tr>
<td>D'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>In</td>
</tr>
</tbody>
</table>
Exercise

There are 6 symmetry transformations for an equilateral triangle in the plane. Make an input-output table for each, and then complete the Cayley table for the group of symmetries of the triangle.