Math Circle Lesson Bowling Pin Puzzle

Jessica Delgado

Introduction

This lesson was given to high school age students during math circles of Fall of 2012. This lesson was the last of four and in the first three lessons the students were able to familiarize themselves with modular arithmetic. The lesson teaches students how to model with algebra as well as how to use modular arithmetic. Since students are not used to adding and subtracting via mod it retrain their brains to think about arithmetic in a new way.

The mod 6 bowling pin puzzle is made up of 3 blocks on the bottom row, 2 blocks in the middle row and 1 block on the top, the blocks are stacked as a pyramid (see figure 1). The pyramid this then filled with numbers. The rule of the pyramid follows from a reverse Pascals Triangle, any block is the sum of the two blocks directly below it mod 6. The goal is to use all the numbers 0, 1, 2, 3, 4, 5 exactly once such that they satisfy the rule of the pyramid. An example but not a solution is shown below (see figure 2):

Pre-requisite knowledge

Since this was the last lesson of four on modular arithmetic the students were already comfortable with how mod worked. The students must have a working knowledge of modular arithmetic, especially addition. Mod can be thought of as a clock with 0 to \( a - 1 \) on its face where numbers "wrap around" upon reaching our modulus \( a \). The image below illustrates mod 26. Looking at figure 3 below we can see, \( 29 \equiv 3(\text{mod}26) \) and \( -3 \equiv 23(\text{mod}26) \). Mod can also be thought of as the remainder after dividing a number by the modulus. For example if we want to find \( 89(\text{mod}5) \) we can first divide 89 by 5 and then find the remainder. In this case
the remainder is 4 so therefore $89 \equiv 4 \pmod{5}$. Another instance $7 \equiv 1 \pmod{2}$ because $7/2 = 3$ remainder 1. Another example would be $13 \equiv 3 \pmod{10}$ because $13/10 = 1$ remainder 3.

**Learning Objectives**

1. Students become comfortable using mod 6 and mod 10
2. Students learn how to apply algebra to puzzles
3. Students will learn a new set of problem solving skills
4. Learning how to apply what is learned in small examples to bigger problems

**Materials needed**

1. Adults preferably one per every 5 students
2. Blank Paper
3. Pencils
Instructional plan

The lesson plan is broken up into three work times and three discussions. The amount of time spent during the work times can vary depending on the students.

Work time 1

Have the students attempt the puzzle for about 15 minutes in their groups.

Discussion 1

After about 15 minutes have the students come together for the first discussion.

Question 1: Do you think it's possible?

Question 2: What are some of the observations that you made?
Take at least one observation from each group and write them on the board. Also ask questions to make sure some observations are realized such as 0 must be at the top or that the two numbers in the middle row must be additive inverses of each other.

Work time 2

This work time can be done together or in the individual groups with each adult. From experience the students will need guidance in order to solve this next part.
Suppose instead of trying the puzzle brute force we used algebra and the rules of mod 6 instead. We know 0 must be at the top which will make the next row additive inverses of each other, we will call these b and -b. On the bottom row we will let the bottom left block be a and then by rules of the triangle the next block must be \(-a + b\) and the block to the right of that must be \(a - 2b\).

\[\begin{array}{c}
0 \\
b \\
a \neg b \\
\end{array}\]

\[\begin{array}{c}
b \\
a - 2b \\
a \\
\end{array}\]

If we add up all the blocks we get \(5 + 4 + 3 + 2 + 1 + 0 = (5 + 0) + (4 + 1) + (3 + 2) = 5 \cdot 3 = 15 \equiv 3 \text{ mod } 6\).

Now before we add up all the \(a\)’s and \(b\)’s we can see some cancellations will occur which will make our work much easier. Below each cancellation is color coded to match its additive inverse.
This leaves us with the equation \( a - b = 3 \). Solving for \( b \) gives \( b = a - 3 \) which is equivalent to \( b = a + 3 \) because \(-3 \equiv 3 \pmod{6} \). Substituting \( b = a + 3 \) into the pyramid gives us:

Notice that again we can change \(-3\) to \(3\) in the right block on the second row. Now the students should be able to substitute the values \(\{1, 2, 4, 5\}\) in for \(a\) to obtain 4 different solutions if we count reflections. Give the above information in pieces and let the students work out each piece themselves, they may even be able to solve the puzzle algebraically on their own.

**Discussion 2**

Come together again and ask the following questions. Question 1: Where you able to solve the puzzle why or why not?

Question 2: What are some observations you had?

Question 3: There is also a similar puzzle with 10 blocks \( \pmod{10} \), do you think these observations would apply to that puzzle as well? (they do)

![Figure 4: solution a=1](image)
If time permits we will set up the mod 10 puzzle and let them try to solve it using the observations they made with the mod 6 puzzle.

**Reflection**

This lesson plan has been implemented in three different classrooms so far. The first setting the students were very fast and all I had to do was ask questions, very little guidance was needed. Almost all of the students were able to get one solution in the first work time. When I prompted that there may be more a majority of students were able to find the remaining solutions on their own. Only when I started going
through the algebra explanation of how to solve the puzzle did students need to follow along. Looking back I should have let each group go at its own pace. There were groups already well into solving the mod 10 puzzle while I was explaining the algebra for the mod 6. Even though most students were able to solve the mod 6 puzzle quickly they were still engaged and interested in the algebra. After they solved the mod 6 they were consumed with how to do the mod 10 puzzle. The speed of this class was not the norm compared to the other classrooms we taught this lesson in.

The other two classrooms were very similar in speed and this lesson plan was perfect for both of their speeds. Only one or two students were able to get a solution during the first work time but everyone else was able to contribute observations. We wrote all these observations, gave some more hints and then let them work again. After some more time we went through the algebraic solution together, letting the students do most the work while we asked the questions. All the students were engaged during this time and everyone was contributing. When we got the algebra solution some students were surprised then when they plugged in either \{1, 2, 4, 5\} for \(a\) that the puzzle was solved. In both of these classrooms this was the major time spent and there was only about 5-10 minutes left for students to apply what they learned to the mod 10 puzzle.

I believe this is a good way for students to get familiar with modular arithmetic, all the students so far have been engaged with the activity. Also not only have all the students been engaged but they had fun and really enjoyed doing the puzzle. In most high school math classrooms the students do not learn by doing puzzles or playing games. The reason this lesson works, I believe, is that the students think they are playing a game. This lesson combines algebra, linear equations and modular arithmetic and disguises it under the veil of a puzzle.