Explicit Descriptions of Surfaces

1. Pictures/diagrams

2. Subspace of \( \mathbb{R}^n \)
   - Level set of some function
     \[ f(x, y, z) = x^2 + y^2 + z^2 \]
     \[ S^2 = f^{-1}(1) \]
   - Image of a function
     \[ (\theta, \phi) \rightarrow (r \cos \theta, r \sin \theta, 0) \]
     \[ + (r \cos \theta \cos \phi, r \sin \theta \cos \phi, r \sin \phi) \]

3. Cut & Paste operations / Normal forms

Gluing maps: \( \Delta \rightarrow \text{Spec} \mathbb{C} \rightarrow S^2 \)
Connected Sums

\[
\begin{align*}
A & \quad + \quad A' \\
\Rightarrow & \quad B \quad + \quad B' \\
\Rightarrow & \quad \text{genus 2} \\
\Rightarrow & \quad \text{genus g} \\
\Rightarrow & \quad 4g - g_{\text{on}} \quad \text{identifiers使命}
\end{align*}
\]

Classification of Surfaces
- Reducing genus
- Simply-connectedness

Theorem Every simply-connected surface is homeomorphic to \( S^2 \) or \( \mathbb{R}^2 \)

(With the resolution of Poincaré conjecture, we are now optimistic about classifying all 3-manifolds. (closed))

Role of simply-connected Surfaces spaces
Given any manifold \( M \), there is an associated simply-connected manifold \( \tilde{M} \) (called its universal cover) of the same dimension and a map \( p: \tilde{M} \to M \) such that

\( M \) can be viewed as a quotient space of \( \tilde{M} \) by a group \( \pi_1(M) \) acting on \( M \).
Example $\mathbb{M} = T^2$ $\tilde{\mathbb{M}} = \mathbb{R}^2$ $\pi_1(\mathbb{M}) = \mathbb{Z}^2$

$\tilde{\mathbb{M}}$ is called the universal cover of $\mathbb{M}$

$\pi_1(\mathbb{M})$ is called the fundamental group of $\mathbb{M}$

Example $\mathbb{M} = T^n$, $\tilde{\mathbb{M}} = \mathbb{R}^n$ $\pi_1(\mathbb{M}) = \mathbb{Z}^n$

Example $\mathbb{M} = \mathbb{RP}^2$ $\tilde{\mathbb{M}} = S^2$ $\pi_1(\mathbb{M}) = \mathbb{Z}/2\mathbb{Z}$

General philosophy: Represent surfaces (or manifolds) as quotients of a "geometry" by a group of discrete group of isometries $(\tilde{\mathbb{M}}, \tilde{\pi})$.

$S^2$ $\leftarrow \mathbb{R}^2$ $\Rightarrow$ $\mathbb{H}^2$

- $K = 1$
- $K = 0$
- $K = -1$

Q: What is curvature?

- $K = \frac{1}{\mathbb{R}}$
- $K = \frac{-1}{\mathbb{R}}$

$Z = x^2 - y^2$

Not constant
algebraic curves sitting inside $\mathbb{CP}^2$ have true and -ve curvature

Tractrix

"pseudo-sphere"

$K = -1$

Hyperbolic plane $H^2$

$ds = \frac{ds_{eucl}}{y}$

$H^2 = \{ (x, y) : y > 0 \}$

length $(L_1) = \sqrt{3}$

length $(L_2) = \sqrt{\frac{1}{2}}$

Geodesics (straight lines) are vertical lines or circles

1 to the x-axis.

Model for tractrix