(1) Suppose $V$ and $W$ are finite-dimensional vector spaces, $T \in L(V,W)$, and define $\operatorname{rank}(T):= \dim \operatorname{range}(T)$. Prove that if $\operatorname{rank}(T) = \dim W$ then $T$ is surjective.

Proof. Suppose $\operatorname{rank}(T) = \dim W$. We know that $\operatorname{range}(T)$ is a subspace of $W$, so consider a basis of $\operatorname{range}(T)$, which we can then extend to a basis of $W$. But since $\operatorname{rank}(T) = \dim W$ these bases must have the same length, and so the basis of $\operatorname{rank}(T)$ is already a basis of $W$, whence $\operatorname{range}(T) = W$. □

(2) Suppose $U$, $V$, and $W$ are finite-dimensional vector spaces, $S \in L(U,V)$, and $T \in L(V,W)$.

(a) Show that $\operatorname{rank}(TS) \leq \min(\operatorname{rank}(S), \operatorname{rank}(T))$. Give examples to show that both equality and strict inequality are possible. Can two nonzero maps have composition equal to zero?

(b) If $S$ has a right inverse, show that $\operatorname{rank}(TS) = \operatorname{rank}(T)$.

(c) If $T$ has a left inverse, show that $\operatorname{rank}(TS) = \operatorname{rank}(S)$.

(d) Prove that the rank of a matrix (i.e., the rank of the underlying linear map) is not more than the number of rows or the number of columns of the matrix.

(e) A matrix is said to have maximal rank if the rank is equal to the the minimum of the number of rows and the number of columns. Show that a matrix has maximal rank if and only it is either injective or surjective.

Proof. (a) It suffices to show that $\operatorname{rank}(TS) \leq \operatorname{rank}(S)$ and $\operatorname{rank}(TS) \leq \operatorname{rank}(T)$. The latter follows directly from $\operatorname{range}(TS) \subseteq \operatorname{range}(T)$ (which we know by adapting the previous homework #4). To prove the former, note that

$$\dim U = \dim \operatorname{null}(TS) + \operatorname{rank}(TS) \quad \text{and} \quad \dim U = \dim \operatorname{null}(S) + \operatorname{rank}(S).$$

Again by adaptation of the previous homework #4, we know that $\operatorname{null}(TS) \subseteq \operatorname{null}(S)$ and thus $\operatorname{rank}(S) \geq \operatorname{rank}(TS)$. An example that shows that $\operatorname{rank}(TS) = \min(\operatorname{rank}(S), \operatorname{rank}(T))$ is possible is given by $S = T =$ identity map on an arbitrary vector space $U = V = W$. An example that shows that $\operatorname{rank}(TS) < \min(\operatorname{rank}(S), \operatorname{rank}(T))$ is possible is given by $U = V = W = \mathcal{P}_2(\mathbb{R})$ and $S = T = \frac{d}{dx}$.

(b) Suppose $SR$ is the identity map. By using part (a) twice,

$$\operatorname{rank}(T) \geq \operatorname{rank}(TS) \geq \operatorname{rank}(TSR) = \operatorname{rank}(T),$$

and so we must have equality all around. (One can also argue via surjectivity of $S$.)

(c) Suppose $LT$ is the identity map. By using part (a) twice,

$$\operatorname{rank}(S) \geq \operatorname{rank}(TS) \geq \operatorname{rank}(LTS) = \operatorname{rank}(S).$$

(One can also argue via injectivity of $T$.)

(d) The rank of a linear map $M$ is the dimension of $\operatorname{range}(M)$, which can be at most the dimension of the codomain of $M$, and that is the number of rows of the matrix corresponding to $M$. By our dimension–null space–rank theorem, the rank of $M$ is also at most the dimension of the domain of $M$, which equals the number of columns of the matrix corresponding to $M$.

(e) Suppose the linear map underlying $M$ is from $V$ to $W$. Then

$$\operatorname{rank}(M) = \# \text{ rows of } M \iff \dim \operatorname{range}(M) = \dim W \iff \operatorname{range}(M) = W,$$

i.e., $M$ is surjective, and

$$\operatorname{rank}(M) = \# \text{ columns of } M \iff \dim \operatorname{range}(M) = \dim V \iff \dim \operatorname{null}(M),$$

i.e., $M$ is injective. □

(3) Consider the linear operator $\frac{d}{dx} \in L(\mathcal{P}_n(\mathbb{F}))$ given by differentiation. Compute the matrix of $\frac{d}{dx}$ using the basis

(a) $1, x, x^2, \ldots, x^n$;

(b) $\left(\begin{array}{c}1 \\ 0 \end{array}\right), \left(\begin{array}{c}1 \\ 1 \end{array}\right), \ldots, \left(\begin{array}{c}1 \\ n \end{array}\right)$. 
Suppose \( \frac{d}{dx} x^k = kx^{k-1} \), the matrix of \( \frac{d}{dx} \) with respect to the monomial basis has entries 
\( a_{k-1,k} = k \) for \( 2 \leq k \leq n \) and \( a_{jk} = 0 \) otherwise.

(b) We claim that
\[
\frac{d}{dx} \left( \binom{x}{k} \right) = \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^{j-1}}{j} \binom{x}{k-j}
\]

which is really a finite sum, as \( \binom{x}{j} = 0 \) when \( j < 0 \) by definition and prove this by induction on \( k \). 
The base case \( k = 0 \) just says \( 0 = 0 \). The induction step follows with
\[
\frac{d}{dx} \left( \binom{x}{k} \right) = \frac{d}{dx} \left( \binom{x}{k-1} + \binom{x}{k} \right)
= \sum_{j=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{j-1}}{j} \binom{x}{k-1-j} + \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^{j-1}}{j} \binom{x}{k-1-j}
= \sum_{j=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{j-1}}{j} \left( \binom{x}{k-j} + \binom{x-1}{k-j-1} \right)
= \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{(-1)^{j-1}}{j} \binom{x}{k-j}.
\]

Thus the matrix of \( \frac{d}{dx} \) with respect to the binomial-coefficient basis has entries
\[
a_{jk} = \begin{cases} 
\frac{(-1)^{j-1}}{k-j} & \text{if } j < k, \\
0 & \text{otherwise}.
\end{cases}
\]

(4) Suppose \( V \) is a finite-dimensional vector space and \( S, T \in \mathcal{L}(V) \). Prove that \( ST \) is invertible if and only if both \( S \) and \( T \) are invertible. Give an example that shows that this statement is not true for infinite-dimensional vector spaces.

Proof. Suppose \( ST \) is invertible and so, in particular, \( \operatorname{null}(ST) = \{0\} \) and \( \operatorname{range}(ST) = V \). We have shown in a previous homework that \( \operatorname{null}(T) \subseteq \operatorname{null}(ST) = \{0\} \), and so \( T \) is injective and (by a theorem from class) invertible. We have also shown that \( \operatorname{range}(S) \supseteq \operatorname{range}(ST) = V \), and so \( S \) is surjective and (by the same theorem) invertible.

An example of how this statement can fail if \( V \) is infinite dimensional is given by \( V = \mathcal{P}(\mathbb{R}) \), \( S(p(x)) = p'(x) \), and \( T(u) = \int_0^1 p(t) \, dt \); by the Fundamental Theorem of Calculus, \( ST \) is the identity map; however, \( S \) is not invertible.

(5) Suppose \( V \) and \( W \) are finite-dimensional vector spaces, and \( U \) is a subspace of \( V \). Let \( R : \mathcal{L}(V,W) \to \mathcal{L}(U,W) \) be the restriction map defined by \( (R(T))(u) = T(u) \).

(a) Show that \( R \) is linear.

(b) Show that \( R \) is surjective.

(c) If \( U \) is a proper subspace of \( V \), show that the restriction map is not injective.

Proof. (a) Given \( S, T \in \mathcal{L}(V,W) \) and \( a \in \mathbb{F} \), we have for any \( u \in U \)
\[
R(aS + T)(u) = aS(u) + T(u) = aR(S)(u) + R(T)(u).
\]

(b) Let \( \{u_1, u_2, \ldots, u_m\} \) be a basis of \( U \), and extend it to a basis \( \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \) of \( V \).

Given \( T \in \mathcal{L}(U,W) \), define \( S \in \mathcal{L}(V,W) \) through
\[
v = \sum_{j=1}^{m} a_j u_j + \sum_{j=1}^{n} b_j v_j \quad \mapsto \quad S(v) = \sum_{j=1}^{m} a_j T(u_j).
\]

This map \( S \) is by definition linear, and \( R(S) = T \). Hence \( R \) is surjective.

(c) Let \( m := \dim U \), \( n := \dim V \), and \( k := \dim W \). Then we know from a theorem in class that \( \dim \mathcal{L}(V,W) = nk \) and \( \dim \mathcal{L}(U,W) = mk \). If \( U \subseteq V \) then \( m < n \), and so \( \operatorname{rank}(R) \) (which is at most \( \dim \mathcal{L}(U,W) \)) cannot be equal \( \dim \mathcal{L}(V,W) \), i.e., \( R \) cannot be surjective.