1. Next, for an arbitrary matrix, I’ll define semantic notions parallel to syntactic consequence and closure.
   a. A formula \( q \in F \) is called a semantic consequence of a set \( S \subseteq F \) of assumptions—written \( S \models q \) —if for every valuation \( \varphi \) related to the matrix, \( \varphi[S] \subseteq T \Rightarrow \varphi(q) \in T \).
   b. That is, the semantic consequences of a set of assumptions are those formulas that have true values whenever all the assumptions do.
   c. The semantic closure of a subset \( S \subseteq F \) is the set \( c'(S) \) of all its semantic consequences.

2. **Theorem.** In fact, \( c' \) is a closure operator on \( F \) in the sense of the Complete Lattices unit. **Proof.**
   a. \( S \subseteq c'(S) \) for any \( S \subseteq F \), obviously.
   b. Therefore \( c'(S) \subseteq c'(c'(S)) \). Next, I’ll prove the reverse inclusion, to get equality here. Suppose \( q \in c'(c'(S)) \). To show that \( q \in c'(S) \), assume that \( \varphi \) is a valuation for which \( \varphi[S] \subseteq T \); I must show that \( \varphi(q) \in T \). The assumption in the previous sentence implies that \( \varphi(r) \in T \) for each \( r \in c'(S) \). According to the first sentence in (i), \( \varphi(q) \in T \) as desired.
   c. If \( R \subseteq S \subseteq F \), then \( c'(R) \subseteq c'(S) \), obviously.

3. The remainder of this discussion is devoted to studying the relationship between syntactic and semantic consequences and closure.

4. The tautologies determined by an implication matrix are the formulas \( p \) such that \( \varphi(p) \in T \) for every valuation \( \varphi \) related to the matrix.

5. **Theorem.** If all the axioms in \( P \) are tautologies with respect to a given implication matrix, then all syntactic consequences of a subset \( S \subseteq F \) are also semantic consequences. That is, \( c(S) \subseteq c'(S) \) **Proof.** Suppose \( S \models q \). Then there is a deduction of \( q \) from \( S \): a sequence of formulas \( p_0, \ldots, p_{n-1} \) such that \( p_{n-1} = q \) and for each \( m < n \),
   a. \( p_m \in S \) —in this case \( p_m \in c'(S) \) by the previous theorem—or
   b. \( p_m \in P \) —in this case \( p_m \in c'(S) \) by hypothesis—or
   c. there exist \( k, l < m \) such that \( p_k = (p_l \Rightarrow p_m) \).
   I’ll show by recursion that \( p_m \in c'(S) \) for each \( m < n \). The recursion hypothesis is that \( p_j \in c'(S) \) for each \( j < m \). To show that \( p_m \in c'(S) \) in case (c), suppose that \( \varphi \) is a valuation for which \( \varphi[S] \subseteq T \). Then \( \varphi(p_j \Rightarrow p_m), \varphi(p_l) \in T \), hence \( \varphi(p_m) \in T \) by (12c).

6. All the axioms adopted so far are familiar tautologies with respect to the material implication matrix. Therefore, all syntactic consequences of a subset \( S \subseteq F \) are also semantic consequences with respect to that matrix. That should not be surprising, since much of the apparatus constructed so far is merely a set-theoretic implementation of familiar informal Boolean theory.

7. Now I’ll introduce a different matrix, to provide a counterexample to the converse of the previous statement.
a. Consider the matrix \(<\mu_B, \Phi, T>\) with
\[
\Phi = \{0, \frac{1}{2}, 1\} \quad T = \{1\} \\
p & q & p \mu_B q \\
1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0
\]
Be sure to check
\[
(\forall p, q \in \Phi)[p \in T \quad \& \quad p \mu_B q \in T \quad \Rightarrow \quad q \in T]\]
\[
\frac{1}{2} & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & 0 & 0 \\
0 & 1 & 1 \\
0 & \frac{1}{2} & 1 \\
0 & 0 & 1
\]

b. It’s tedious but not hard to check that all the axioms adopted so far are tautologies with respect to \(\mu_B\).

c. I could give a very informal argument why that shouldn’t be surprising, but it wouldn’t be very convincing, so I won’t.

d. Therefore, all syntactic consequences of a subset \(S \subseteq F\) are also semantic consequences with respect to that matrix.

e. It’s easy to check that if \(p, q\) are variables, then \((p \Rightarrow q) \Rightarrow p\), often called Peirce’s law, is not a tautology with respect to \(\mu_B\). Thus Peirce’s law is not a syntactic consequence of \(\phi\), with the current axiom set.

f. That shouldn’t be surprising, because both Peirce’s law and \(\mu_B\) seem far-fetched.

g. What is surprising to me is that Peirce’s law is a tautology with respect to the material implication matrix: check it! Thus Peirce’s law is a semantic consequence of \(\phi\) under material implication.

h. Thus, with the current axiom set, \(c'(S) \subseteq c(S)\) in general.

i. This method of showing unprovability, and I think this specific result, was introduced by Paul Bernays in 1919, but published only in 1926.