1. **In class, the Cardinals II unit**
   a. I discussed the proof that $\omega + \omega = \omega$. In particular, it was easy to find a formula for the required bijection from $\mathbb{N}$ to $(\mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{1\})$.
   b. For $n \in \mathbb{N}$, it follows that $\omega \leq n + \omega \leq \omega + \omega = \omega$, so $n + \omega = \omega$. This use of the weak antisymmetry of $\leq$—i.e., the Cantor–Bernstein theorem—is typical. It lets us avoid nasty proofs.
   c. Some presentations of cardinal arithmetic include subtraction. It’s not very interesting. Nevertheless, you can define $\alpha - \beta$ as the cardinal $\delta$ such that $\alpha = \beta + \delta$, provided you can show there is a unique such $\delta$.
   d. Thus if $n \in \mathbb{N}$ then $\omega - n = \omega$.
   e. [Click here](#) for a cartoon. It calls to mind the band-bus chant that starts, “Ninety-nine bottles of beer on the wall, ninety-nine bottles of beer. A bottle of beer fell off the wall, ninety-eight bottles of beer on the wall....” Set theory transcends puerility.
   f. If $\alpha$ is the cardinal of an infinite set $A$ then $\omega + \alpha = \alpha$. **Proof.** We can assume $A$ is disjoint from $\mathbb{N}$. Since $\omega \leq \alpha$, there is an injection $\alpha : \mathbb{N} \to A$. Define $f : \mathbb{N} \cup A \to A$ by setting $f(n) = a_{2n}$ and $f(a_n) = a_{2n+1}$ for each $n \in \mathbb{N}$, and $f(x) = x$ for each $x \in A - \text{Rng } a$.
   g. I discussed the proof that $\omega \omega = \omega$. It’s visually obvious that there is a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, but it would be very hard to write down a formula for the obvious zig-zag function.

2. Martin Davis (1958, 43–45) proved that the formula 
   $$f(<x,y>) = \frac{1}{2}((x + y)^2 + 3x + y)$$
   defines a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and gave formulas for the components of its inverse. I don’t remember whether that has any visual impact.

3. **In class, or in the office**, students and I have discussed routine exercises 1–10, 12 and substantial problems 1, 2. These are either assigned or will be presented in class.

4. Cardinals of infinite well-ordered sets are called *alephs*. This terminology is due to Cantor, who introduced the sequence of cardinals $\aleph_0, \aleph_1, \aleph_2, \ldots$, beginning with the smallest one. Further, he introduced $\aleph_\nu$ for each ordinal $\nu$. According to the well-ordering theorem, which is equivalent to the axiom of choice, every set can be well-ordered, so every cardinal is an aleph. For that reason, the aleph notation is often used for cardinals in general. These notes avoid it because they avoid ordinals. In particular, $\omega$ is used here for the smallest infinite cardinal, in place of Cantor’s $\aleph_0$. 
