1. *Assignment:* Struik, sections 8.1–8.3 (covered today), 8.7 (next).
2. Mr. Shahindoust gave a brief discussion of his first term paper, on Pascal and Leibniz. Both made major contributions to physics and to several areas of mathematics, and in philosophical studies outside mathematics. Their lives barely overlapped; Pascal died young. Mr. Shahindoust compared and contrasted their lives. His treatment of two figures must have been difficult to research and arrange, but it led to a very interesting paper.
3. During lecture 24, I missed showing the photographs of the Turin Academy of Sciences, intended to show the forbidding atmosphere in which Peano's paper was presented. Actually, I had neglected to set the links. Return to item 5b of that lecture, and do that.
4. *What is a function? continued*
   a. I backtracked from Struik's reference and found the original definition, in Lejeune Dirichlet 1837, pages 135–136 in the reprinted edition. Click here for those pages. Dirichlet stressed that there should be no assumption of any explicit rule or formula for determining the function values. Note that this is his introduction to a work about Fourier series. Here is a rough translation:

On the representation of completely arbitrary functions by sine and cosine series

The remarkable series, which in a specified interval represent functions that are free of any law, or which satisfy completely different laws in different parts of this interval, have had so many applications in the analytical treatment of physical problems since the foundation of the mathematical heat by Fourier, that it seems appropriate for the following volumes of this work to introduce particular portions of the most recent studies of various parts of mathematical physics by the development of some of the most important of these series.

§1

We consider a and b as two fixed numbers and x as a variable quantity that gradually takes all values between a and b. Now if to each x there should correspond a unique finite y in such a way that while x proceeds through the interval from a to b continuously, $y = f(x)$ varies gradually, then y is called a *continuous* function of x on this interval. In this it is not at all necessary that y should be specified by the same law in this entire interval; in fact one doesn't even need to consider any law expressible by mathematical operations. Geometrically speaking, that is, regarding x and y as abscissas and ordinates, a continuous function appears as a connected curve, according to which each abscissa between a and b corresponds to only one point. This definition does not ascribe to the various parts of the curve any common law; one can consider it drawn by putting together various parts, or without any law whatever. It follows from this that such a function is to be considered completely defined on an interval only when it is either given in this whole region graphically, or mathematical rules are given for various parts of it. In case one has determined a function only for a part of the interval, the manner of its continuation over the rest of the interval remains completely arbitrary.

5. *Struik, sections 8.1–8.3: Gauss and Legendre*

a. Although Gauss published a lot, much more of his work went unpublished until years after his death.

b. Gauss's name is attached to many key theorems, definitions, and methods covered in my high-school, undergraduate, and graduate courses. Some of these require acknowledgements to Legendre as well. I'll list them almost chronologically.

c. *Complex numbers*. Gauss introduced, or greatly popularized, the use of complex numbers in many areas of mathematics, using the idea of the complex plane. I learned that in high school.

d. *Theory of equations*. Gauss's doctoral thesis contained the first valid proof of the fundamental theorem of algebra: every nonconstant polynomial with real coefficients has a complex root.

   i. That was preached but not proved in high school. The easiest proof is Liouville's, which comes about halfway through a first-semester graduate course in complex analysis. It's very different from Gauss's first proof.

   ii. Since it's easy to show that such a polynomial has only finitely many roots, that the complex roots occur in conjugate pairs $r, \bar{r}$, and that $(x - r)(x - \bar{r})$ is a real quadratic, the fundamental theorem entails that every nonconstant real polynomial is the product of real linear and quadratic factors. That I learned in high school. And it's the title of Gauss 1797, his doctoral thesis, written when he was twenty: *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variablilis in factores reales primi vel secundi gradus resolvi posse*. Click here to see its first page.

e. *Number theory*. Gauss's 1801 *Disquisitiones arithmeticae* collected what was known in that field, and established a foundation and terminology for the use of congruences (modular arithmetic). These included the proof of the quadratic reciprocity theorem that was the capstone of my freshman course in number theory: for odd primes p and q, it tells the precise relationship of the truth of the conditions "p is congruent to a square mod q" and "q is congruent to a square mod p". Today's number theory books use Legendre's approach to this question, too.

f. *Constructibility*. Using number theory and algebra Gauss discovered criteria for constructability of angles $2\pi/n$ by straightedge and compass, foreshadowing more general work by others in the theory of equations. It was previously known that these were constructible for $n = 3, 4, 5, 6, 8, 10, 12,$ and 16, for example, but his construction was new for $n = 17$ and 257 (the next prime that fits his criterion). I was intrigued by such things in high school, but somehow that result never got into my curriculum.

g. *Least-squares*. Gauss was involved in many astronomical and geodesic projects that required precise computational analysis of large amounts of experimental data. He invented or popularized the use of the normal distribution for analyzing errors, and the techniques for least-squares polynomial approximation that you study in calculus and statistics classes. When these polynomials are used to approximate continuous data, they are based on Legendre polynomials. I think I learned about these things in college, in second-year calculus and physics classes.

h. *Surface integrals*. To handle problems in electrical theory Gauss formulated the vector integral theorem named for him: the integral over a body B of the divergence of a vector field F equals the integral over the boundary ∂B of B of the component of F perpendicular to ∂B pointing outward. That's what I learned sophomore year, but I notice that I can hardly recognize the terms used today for this subject.

i. *Linear algebra*. The scientific analyses Gauss undertook required solution of large linear systems. That is, much larger than I ever contemplated in high school. That required organization. If you organize that computation optimally you get the algorithm called *Gauss elimination*. If you're careful, you can derive almost all the main results of linear algebra from its properties. I didn't learn that the first time around, in a junior linear algebra class, but certainly did senior year in a graduate numerical computation class.