Review for the Final Exam
Math 376
Fall, 2014

(1) Find the general solution of the differential equation

$$y'' + y = 3\sin 2t + t\cos 2t$$

(2) Find a particular solution to

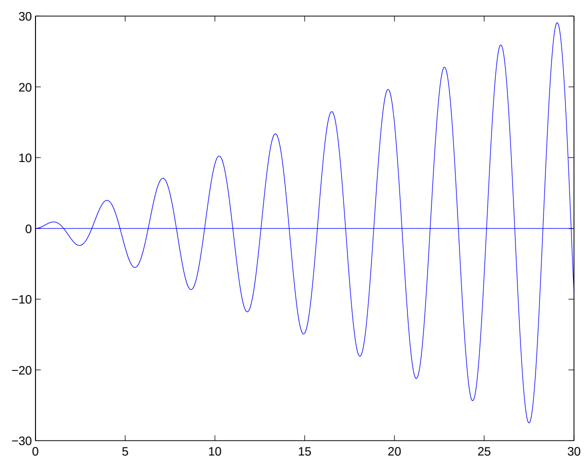
$$y'' - 2y' + y = xe^x + 4x.$$

(3) Verify that $y_1 = 1 + t$, $y_2 = e^t$ are two solutions to the corresponding homogeneous equation of $ty'' - (1 + t)y' + y = t^2 e^{2t}$, $t > 0$. Find the general solution.

(4) The following graph depicts the motion of a mass attached to a vibrating spring. Identify the equation of motion from the following list (numbered (a) - (d)) that is likely accounted for the graph, and identify the phenomenon it represents from the second list

(a) $y'' + 4y = 4\cos 2t$ (b) $y'' + 4y = 4\cos 2.1t$
(c) $y'' + 4y' + 4y = 0$ (d) $y'' + 4y' + 8y = 0$

(i) Damped Oscillation. (ii) Resonance.
(iii) Critically Damped motion. (iv) Beats.



(5) Which of the equations in the (a) - (d) list gives rise to which phenomenon in the second list? How would you have to argument the graph to determine which of the two phenomena (resonance and beats) is really present? Why can you definitely rule out the other two?

(6) Determine the equation of motion (solution) for a system governed by

$$x'' + x = 5\cos\omega t, \quad x(0) = 0, x'(0) = 1,$$

where ω is an unknown constant. Sketch the solution.

(7) Use the method of variation of parameters to show that

$$y(t) = c_1\cos t + c_2\sin t + \int_0^t f(s)\sin(t - s)ds$$

is a general solution to the different equation $y^{''}+y=f(t)$, where $f(t)$ is a continuous function on $(-\infty,\infty)$.

(8) Find the general solution to
$$y^{'''}+y^{''}+y^{'}+y=0.$$

(9) We want to build a system $\mathbf{x}'=A\mathbf{x}$ so that A has eigen-trajectories $2x_2=x_1$ and $x_2=2x_1$ with corresponding eigenvalues -1 and 1, respectively. Find A, and show that every solution curve approaches the line $2x_2=x_1$ as time goes by, and approaches $x_2=2x_1$ as time goes by backwards.

(10) [10] Consider the system
$$\mathbf{x}'=\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}\mathbf{x}.$$
Determine which of the following conditions on b and c ensure that some sol ution has a trajectory along a straight line.

(a) $0<b,c$;
(b) $b<0<c$;
(c) $b,\ c<0$.

(11) In each of the two parts below you are given a matrix A and some information about its eigenvalues and eigenvectors. In each case, consider the system $\mathbf{x}'=A\mathbf{x}$ (the derivative is w.r.t. t). Then sketch several trajectories in the phase plane, indicating the direction of increasing time t, and discuss the type and stability of the origin.

(a) The matrix $A=\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, and
$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}=-\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(b) The matrix $A=\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, and
$$\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=-\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}=-2\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(12) Suppose that $y''(t)=-y(t)$.

(a) Convert the differential equation into a system of first order differential equations by letting $x_1=y$ and $x_2=x_1'$ (so that $x_1'=x_2$ and $x_2'=x_1''$, etc.).

(b) Find the general solution to the system from (a).

(c) Show that the trajectories in the phase plane (x_1x_2 plane) are circles, i.e., to show that the relationship between x_1 and x_2 satisfies a circle equation.

(13) Find the general solution of the given system. Draw a few trajectories, and describe how the solutions behave as $t\to\infty$:
$$\mathbf{x}'=\begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\mathbf{x}.$$