Part I. All HW problems of Chapters 1 - 3, 5.1, 5.2.

Part II. True or False (Some of these problems require calculations)

1. A system of 4 linear equations in three unknowns is always inconsistent.
2. If the 4x4 matrix $A$ has rank 4, then any linear system with the coefficient matrix $A$ will have a unique solution.
3. There exists a 5x5 matrix $A$ of rank 4 such that the system $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.
4. Vector \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\] is a linear combination of vectors \[
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
7 \\
8 \\
9
\end{bmatrix}
\]
5. If $A$ is a nonzero matrix of the form \[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\], then the rank of $A$ must be 2.
6. There exists a 2x2 matrix $A$ such that
   \[
   A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
   \]
7. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.
8. If $\vec{u}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$, and $\vec{v}$ is a linear combination of vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, then $\vec{u}$ must be a linear combination of $\vec{p}$, $\vec{q}$, and $\vec{r}$ and $\vec{w}$.
9. If $A$ is a 4x3 matrix of rank 3 and $A\vec{v} = A\vec{w}$ for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$, then $\vec{v} = \vec{w}$.
10. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is a linear transformation.
11. Matrix \[
\begin{bmatrix}
1/2 & -1/2 \\
1/2 & 1/2
\end{bmatrix}
\] represents a rotation.
12. If $AB = I_n$ for two $n \times n$ matrices $A$ and $B$, the $A$ must be the inverse of $B$.
13. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.
14. Matrix \[
\begin{bmatrix}
-0.68 & 0.8 \\
-0.8 & -0.6
\end{bmatrix}
\] represents a rotation.
15. \[
\begin{bmatrix}
1 & k \\
0 & 1
\end{bmatrix}^3 = \begin{bmatrix}
1 & 3k \\
0 & 1
\end{bmatrix}
\] for all real numbers $k$.
16. If $A^2 = I_n$, then $A$ (matrix) must be invertible.
17. The equation \( A^{-1} = A \) holds for all 2x2 matrices \( A \) representing a reflection.
18. There exist a 2x3 matrix \( A \) and a 3x2 matrix \( B \) such that \( AB = I_2 \).
19. There exist a 3x2 matrix \( A \) and a 2x3 matrix \( B \) such that \( AB = I_3 \).
20. If the linear system \( A^2 \vec{x} = \vec{b} \) is consistent, then the system \( A \vec{x} = \vec{b} \) is consistent.
21. If \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) are linearly independent vectors in \( \mathbb{R}^n \), then they must form a basis of \( \mathbb{R}^n \).
22. The kernel of any invertible matrix consists of the zero vector only.
23. If \( 2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w} \), then vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) must be linearly dependent.
24. The column vectors of a 5x4 matrix must be linearly dependent.
25. If the kernel of a matrix \( A \) consists of the zero vector only, then the column vectors of \( A \) must be linearly independent.
26. If \( \vec{u}, \vec{v} \) and \( \vec{w} \) are in a subspace \( V \) of \( \mathbb{R}^n \), then the vector \( 2\vec{u} + 3\vec{v} + 4\vec{w} \) must be in \( V \) as well.
27. If vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \) are linearly independent, then vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent.
28. Matrix \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) is similar to \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
29. If a 2x2 matrix \( P \) represents the orthogonal projection onto a line in \( \mathbb{R}^2 \), then \( P \) must be similar to the matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).
30. If vectors \( \vec{u}, \vec{v} \) and \( \vec{w} \) are linearly dependent, then vector \( \vec{w} \) must be a linear combination of \( \vec{u} \) and \( \vec{v} \).
31. If \( A \) and \( B \) are invertible matrices of size \( n \times n \), then \( AB \) must be similar to \( BA \).
32. If \( AB = 0 \) for two 2x2 matrices \( A \) and \( B \), then \( BA \) must be the zero matrix as well.
33. \( \mathbb{R}^2 \) is a subspace \( \mathbb{R}^3 \).
34. For every subspace \( V \) of \( \mathbb{R}^3 \), there exists a 3x3 matrix \( A \) such that \( V = \text{Im}(A) \).
35. If the kernel of a 5x4 matrix \( A \) consists of the zero vector alone, and if \( AB = AC \) for two 4x5 matrix \( B \) and \( C \), the matrices \( B \) and \( C \) must be equal.
36. There is a subspace \( V \) of \( \mathbb{R}^5 \) such that \( \text{dim}(V) = \text{dim}(V^\perp) \), where \( V^\perp \) denotes the orthogonal complement of \( V \).
37. Every invertible \( n \times n \) matrix \( A \) can be expressed as the product of an orthogonal matrix (columns of the matrix forms an ONB of \( \mathbb{R}^n \)) and an upper triangular matrix.
38. If \( \vec{x} \) and \( \vec{y} \) are two vectors in \( \mathbb{R}^n \), then the equation \( ||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 \) must hold.
39. Let \( \vec{v}_1, \ldots, \vec{v}_m \) be a basis of \( V \). A vector \( \vec{x} \) in \( \mathbb{R}^n \) is orthogonal to \( V \) if and if \( \vec{x} \) is orthogonal to all the vectors \( \vec{v}_1, \ldots, \vec{v}_m \).
40. Calculate: find the length of the vector \( \vec{x} = 7\vec{u}_1 + \vec{u}_2 - 4\vec{u}_3 - 2\vec{u}_4 + \vec{u}_5 \), where \( \vec{u}_1, \ldots, \vec{u}_5 \) are orthonormal vectors in \( \mathbb{R}^{10} \).
41. Among all vectors of \( \mathbb{R}^n \) whose components add up to 1, find the vector of minimal length.
42. Can you find a line \( L \) in \( \mathbb{R}^n \) and a vector \( \vec{x} \) in \( \mathbb{R}^n \) such that \( \vec{x} \cdot \text{Proj}_L \vec{x} < 0 \)? Why?
43. If \( A \) has orthogonal columns, but not orthonormal. What are the \( Q \) and \( R \) in the QR-factorization?