A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{vmatrix} \longrightarrow \begin{vmatrix} x \quad - \; z = \;\; 2 \\ y + 2z = -1 \\ 0 = -6 \end{vmatrix}.$$

Whatever values we choose for x, y, and z, the equation $0 = -6$ cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

EXERCISES 1.1

GOAL *Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.*

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\begin{vmatrix} x + 2y = 1 \\ 2x + 3y = 1 \end{vmatrix}$

2. $\begin{vmatrix} 4x + 3y = 2 \\ 7x + 5y = 3 \end{vmatrix}$

3. $\begin{vmatrix} 2x + 4y = 3 \\ 3x + 6y = 2 \end{vmatrix}$

4. $\begin{vmatrix} 2x + 4y = 2 \\ 3x + 6y = 3 \end{vmatrix}$

5. $\begin{vmatrix} 2x + 3y = 0 \\ 4x + 5y = 0 \end{vmatrix}$

6. $\begin{vmatrix} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{vmatrix}$

7. $\begin{vmatrix} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{vmatrix}$

8. $\begin{vmatrix} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{vmatrix}$

9. $\begin{vmatrix} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{vmatrix}$

10. $\begin{vmatrix} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{vmatrix}$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\begin{vmatrix} x - 2y = 2 \\ 3x + 5y = 17 \end{vmatrix}$

12. $\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 6 \end{vmatrix}$

13. $\begin{vmatrix} x - 2y = 3 \\ 2x - 4y = 8 \end{vmatrix}$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{vmatrix}$

15. $\begin{vmatrix} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{vmatrix}$

16. $\begin{vmatrix} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{vmatrix}$

17. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y = a \\ 3x + 5y = b \end{vmatrix},$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\begin{vmatrix} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{vmatrix},$$

where a, b, and c are arbitrary constants.

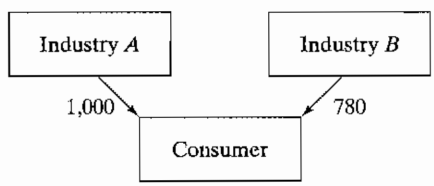
19. Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2, are given by

$$D_1 = 70 - 2P_1 + P_2, \quad S_1 = -14 + 3P_1,$$
$$D_2 = 105 + P_1 - P_2, \quad S_2 = -7 + 2P_2.$$

a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?

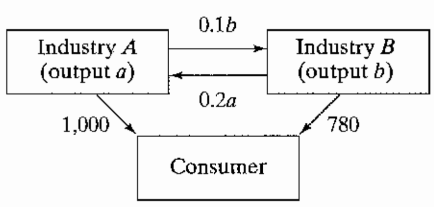
b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.

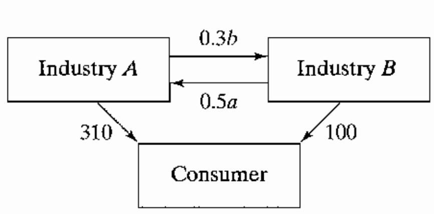


What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?

Yon may be tempted to say 1,000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A prodnces electricity. Of conrse, producing almost any product will require electric power. Suppose that industry B needs 10¢worth of electricity for each \$1 of output B produces and that industry A needs 20¢worth of B's prodncts for each \$1 of outpnt A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.



21. Find the outpnts a and b needed to satisfy the consumer and interindustry demands given in the following figure (see Exercise 20):



22. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t).$$

This equation could describe a forced damped oscillator, as we will see in Chapter 9. We are told that the differential equation has a solution of the form

$$x(t) = a\sin(t) + b\cos(t).$$

Find a and b, and graph the solution.

23. Find all solutions of the system

$$\left| \begin{array}{l} 7x - y = \lambda x \\ -6x + 8y = \lambda y \end{array} \right|, \quad \text{for}$$

a. $\lambda = 5$ **b.** $\lambda = 10$, and **c.** $\lambda = 15$.

24. On your next trip to Switzerland, you should take the scenic boat ride from Rheinfall to Rheinau and back. The trip downstream from Rheinfall to Rheinau takes 20 minutes, and the return trip takes 40 minutes; the distance between Rheinfall and Rheinau along the river is 8 kilometers. How fast does the boat travel (relative to the water), and how fast does the river Rhein flow in this area? You may assume both speeds to be constant throughout the journey.

25. Consider the linear system

$$\left| \begin{array}{rl} x + y - z &= -2 \\ 3x - 5y + 13z &= 18 \\ x - 2y + 5z &= k \end{array} \right|,$$

where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have?

c. Find all solutions for each value of k.

26. Consider the linear system

$$\left| \begin{array}{rl} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (k^2 - 5)z &= k \end{array} \right|,$$

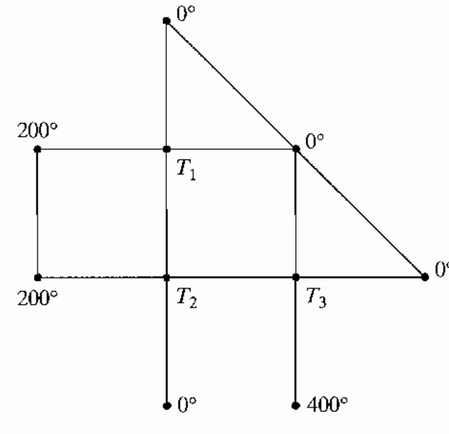
where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which valne(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

27. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

28. In a grid of wires, the temperatnre at exterior mesh points is maintained at constant values (in °C) as shown in the accompanying figure. When the grid is in thermal equilibrium, the temperature T at each interior mesh point is the average of the temperatures at the four adjacent points. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

Find the temperatures T_1, T_2, and T_3 when the grid is in thermal equilibrium.



29. Find the polynomial of degree 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, -1)$, $(2, 3)$, and $(3. 13)$. Sketch the graph of this polynomial.

30. Find a polynomial of degree ≤ 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, p)$, $(2, q)$, $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

31. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 3)$ and $(2, 6)$, such that $f'(1) = 1$ [where $f'(t)$ denotes the derivative].

32. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(2, 0)$, such that $\int_1^2 f(t)\,dt = -1$.

33. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 1$.

34. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 3$.

35. Find the function $f(t)$ of the form $f(t) = ae^{3t} + be^{2t}$ such that $f(0) = 1$ and $f'(0) = 4$.

36. Fiud the function $f(t)$ of the form $f(t) = a\cos(2t) + b\sin(2t)$ snch that $f''(t)+2f'(t)+3f(t) = 17\cos(2t)$. (This is the kind of differential equation you might have to solve when dealing with forced damped oscillators, in physics or engineering.)

37. Find the circle that runs through the points $(5, 5)$, $(4, 6)$, and $(6, 2)$. Write your equation in the form $a + bx + cy + x^2 + y^2 = 0$. Find the center and radius of this circle.

38. Find the ellipse eentered at the origin that runs through the points $(1, 2)$, $(2, 2)$, and $(3, 1)$. Write yonr equation in the form $ax^2 + bxy + cy^2 = 1$.

39. Find all points (a, b, c) in space for which the system

$$\left| \begin{aligned} x + 2y + 3z &= a \\ 4x + 5y + 6z &= b \\ 7x + 8y + 9z &= c \end{aligned} \right|$$

has at least one solution.

40. Linear systems are particularly easy to solve when they are in triangular form (i.e., all entries above or below the diagonal are zero).

a. Solve the lower triangular system

$$\left| \begin{array}{rcl} x_1 & = & -3 \\ -3x_1 + x_2 & = & 14 \\ x_1 + 2x_2 + x_3 & = & 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 & = & 33 \end{array} \right|$$

by forward substitution, finding x_1 first, then x_2, then x_3, and finally x_4.

h. Solve the upper triangular system

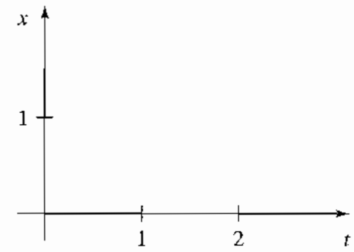
$$\left| \begin{array}{rcl} x_1 + 2x_2 - x_3 + 4x_4 & = & -3 \\ x_2 + 3x_3 + 7x_4 & = & 5 \\ x_3 + 2x_4 & = & 2 \\ x_4 & = & 0 \end{array} \right|.$$

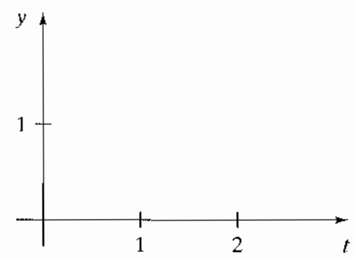
41. Consider the linear system

$$\left| \begin{aligned} x + y &= 1 \\ x + \frac{t}{2}y &= t \end{aligned} \right|,$$

where t is a nonzero constant.

a. Determine the x- and y-intercepts of the lines $x+y = 1$ and $x + (t/2)y = t$; sketch these lines. For which values of the constant t do these lines intersect? For these values of t, the point of intersection (x, y) depends on the choice of the constant t; that is, we can consider x and y as fnnctions of t. Draw rough sketches of these functions.





Explain briefly how yon found these grapbs. Argue geometrically, without solving the system algebraically.

b. Now solve the system algebraically. Verify that the graphs you sketched in part (a) are compatible with your algebraic solution.

42. Find a system of linear equations with three nnknowns whose solutions are the points on the line through $(1, 1, 1)$ and $(3, 5, 0)$.

43. Find a system of liuear equations with three nnknowns x, y, z whose solutions are

$$x = 6 + 5t, \quad y = 4 + 3t, \quad \text{and} \quad z = 2 + t,$$

where t is an arbitrary constant.

44. Boris and Marina are shopping for chocolate bars. Boris observes, "If I add half my money to yours, it will be enough to buy two chocolate bars." Marina naively asks, "If I add half my money to yours, how many can we buy?" Boris replies, "One chocolate bar." How much money did Boris have? (From Yuri Chernyak and Robert Rose, *The Chicken from Minsk*, Basic Books, 1995.)

45. Here is another method to solve a system of linear equations: Solve one of the equations for one of the variables, and snbstitute the result into the other equations. Repeat this process until you run out of variables or equations. Consider the example discussed on page 2:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right| .$$

We can solve the first equation for x:

$$x = 39 - 2y - 3z.$$

Then we substitute this equation into the other equations:

$$\left| \begin{array}{l} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{array} \right| .$$

We can simplify:

$$\left| \begin{array}{l} y - z = -5 \\ -4y - 8z = -91 \end{array} \right| .$$

Now, $y = z - 5$, so that $-4(z - 5) - 8z = -91$, or

$$-12z = -111.$$

We find that $z = \frac{111}{12} = 9.25$. Then

$$y = z - 5 = 4.25,$$

and

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the same as the method discussed in this sectiou; only the bookkeeping is differeut.

46. A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yognrt contains 12 grams of protein and 20 grams of carbohydrates.

a. If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?

b. If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how mauy servings of each item should he consume?

47. I have 32 hills in my wallet, in the denominations of US$ 1, 5, and 10, worth \$100 in total. How many do I have of each denomination?

48. Some parking meters in Milan, Italy, accept coins in the denominations of 20¢, 50¢, and €2. As an incentive program, the city administrators offer a big reward (a brand new Ferrari Testarossa) to any meter maid who brings back exactly 1,000 coins worth exactly €1,000 from the daily rounds. What are the odds of this reward being claimed anytime soon?

1.2 Matrices, Vectors, and Gauss–Jordan Elimination

When mathematicians in ancient China had to solve a system of simultaneous linear equations such as[4]

$$\left| \begin{array}{l} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{array} \right| ,$$

they took all the numbers involved in this system and arranged them in a rectangular pattern (*Fang Cheng* in Chinese), as follows:[5]

3	21	−3	0
−6	−2	−1	62
2	−3	8	32

All the information about this system is conveniently stored in this array of numbers.

The entries were represented by bamboo rods, as shown below; red and black rods stand for positive and negative numbers, respectively. (Can you detect how this

[4]This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*; see page 1. Our source is George Gheverghese Joseph, *The Crest of the Peacock, Non-European Roots of Mathematics*, 2nd ed., Princeton University Press, 2000.

[5]Actually, the roles of rows and columns were reversed in the Chinese representation.

squares, which he had developed around 1794. (See Section 5.4.) Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Gauss later described his methods of orbit computation in his book *Theoria Motus Corporum Coelestium* (1809).

The method of solving a linear system by Gauss–Jordan elimination is called an *algorithm*.[10] An algorithm can be defined as "a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete Steps, 1, 2, 3, ..., whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end" (David Berlinski, *The Advent of the Algorithm: The Idea that Rules the World*, Harcourt Inc., 2000).

Gauss–Jordan elimination is well suited for solving linear systems on a computer, at least in principle. In practice, however, some tricky problems associated with roundoff errors can occur.

Numerical analysts tell us that we can reduce the proliferation of roundoff errors by modifying Gauss–Jordan elimination, employing more sophisticated reduction techniques.

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix A into a matrix B by a sequence of elementary row operations and if B is in reduced row-echelon form, is it necessarily true that $B = \text{rref}(A)$? Fortunately (and perhaps surprisingly) this is indeed the case.

In this text, we will not utilize this fact, so there is no need to present the somewhat technical proof. If you feel ambitious, try to work out the proof yourself after studying Chapter 3. (See Exercises 3.3.84 through 3.3.87.)

[10] The word *algorithm* is derived from the name of the mathematician al-Khowarizmi, who introduced the term *algebra* into mathematics. (See page 1.)

EXERCISES 1.2

GOAL *Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.*

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work. Solve the system in Exercise 8 for the variables x_1, x_2, x_3, x_4, and x_5.

1. $\left| \begin{array}{l} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{array} \right|$

2. $\left| \begin{array}{l} 3x + 4y - z = 8 \\ 6x + 8y - 2z = 3 \end{array} \right|$

3. $x + 2y + 3z = 4$

4. $\left| \begin{array}{l} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{array} \right|$

5. $\left| \begin{array}{l} x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_4 = 0 \end{array} \right|$

6. $\left| \begin{array}{l} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{array} \right|$

7. $\left| \begin{array}{l} x_1 + 2x_2 \quad 2x_4 + 3x_5 = 0 \\ x_3 + 3x_4 + 2x_5 = 0 \\ x_3 + 4x_4 - x_5 = 0 \\ x_5 = 0 \end{array} \right|$

8. $\left| \begin{array}{l} x_2 + 2x_4 + 3x_5 = 0 \\ 4x_4 + 8x_5 = 0 \end{array} \right|$

9. $\left| \begin{array}{l} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{array} \right|$

10. $\left| \begin{array}{l} 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 = -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \end{array} \right|$

11. $\left| \begin{array}{l} x_1 + 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{array} \right|$

12. $$\left|\begin{array}{l} 2x_1 - 3x_3 + 7x_5 + 7x_6 = 0 \\ -2x_1 + x_2 + 6x_3 - 6x_5 - 12x_6 = 0 \\ x_2 - 3x_3 + x_5 + 5x_6 = 0 \\ -2x_2 + x_4 + x_5 + x_6 = 0 \\ 2x_1 + x_2 - 3x_3 + 8x_5 + 7x_6 = 0 \end{array}\right|$$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $$\left|\begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array}\right|$$

14. $$\left|\begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array}\right|$$

15. $$\left|\begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array}\right|$$

16. $$\left|\begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array}\right|$$

17. $$\left|\begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array}\right|$$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

23. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

24. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

25. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. (See Exercise 24.)

26. Consider an $n \times m$ matrix A. Can you transform $\operatorname{rref}(A)$ into A by a sequence of elementary row operations? (See Exercise 25.)

27. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

28. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

29. *Balancing a chemical reaction.* Consider the chemical reaction

$$a\ NO_2 + b\ H_2O \rightarrow c\ HNO_2 + d\ HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

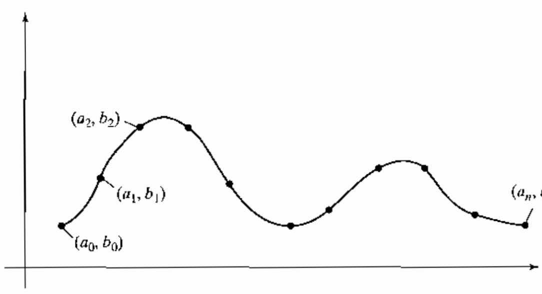
$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

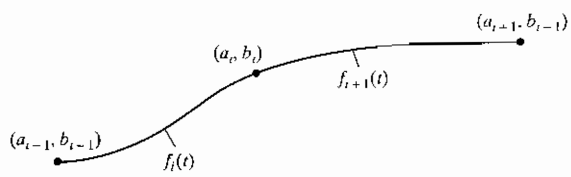
30. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points (0, 1), (1, 0), (−1, 0), and (2, −15). Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points (1, 1), (2, −1), (3, −59), (−1, 5), and (−2, −29). Graph this polynomial.

32. *Cubic splines.* Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points (a_i, b_i) are given to you, and your job is to connect the dots in a reasonably smooth way. Let $a_{i+1} > a_i$.



One method often employed in such design problems is the technique of cubic splines. We choose $f_i(t)$, a polynomial of degree ≤ 3, to define the shape of the ride between (a_{i-1}, b_{i-1}) and (a_i, b_i), for $i = 1, \ldots, n$.



Obviously, it is required that $f_i(a_i) = b_i$ and $f_i(a_{i-1}) = b_{i-1}$, for $i = 1, \ldots, n$. To guarantee a smooth ride at the points (a_i, b_i), we want the first and the second derivatives of f_i and f_{i+1} to agree at these points:

$$f_i'(a_i) = f_{i+1}'(a_i) \quad \text{and}$$
$$f_i''(a_i) = f_{i+1}''(a_i), \quad \text{for } i = 1, \ldots, n-1.$$

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

$$f_1'(a_0) = f_n'(a_n) = 0.$$

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (*Note*: It can be shown that this system has a unique solution.)

33. Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$, where $f'(t)$ is the derivative of $f(t)$. Graph this polynomial.

34. The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$.

Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

35. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

(See Exercise 34.)

36. Find all solutions x_1, x_2, x_3 of the equation

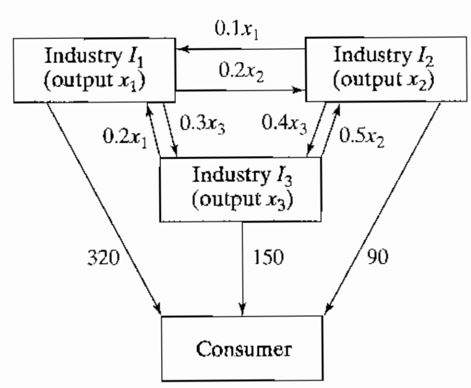
$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input–output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have the industries $I_1, I_2, \ldots, I_n$, with outputs $x_1, x_2, \ldots, x_n$. The *output vector* is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The *consumer demand vector* is

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where b_i is the consumer demand on industry I_i. The demand vector for industry I_j is

$$\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢worth of products from industry I_3 for each \$1 worth of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j\vec{v}_j$?

c. What is the meaning in economic terms of $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}$?

d. What is the meaning in economic terms of the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}?$$

39. Consider the economy of Israel in 1958.[11] The three industries considered here are

I_1: agriculture,
I_2: manufacturing,
I_3: energy.

Outputs and demands are measured iu millions of Israeli pounds, the currency of Israel at that time. We are told that

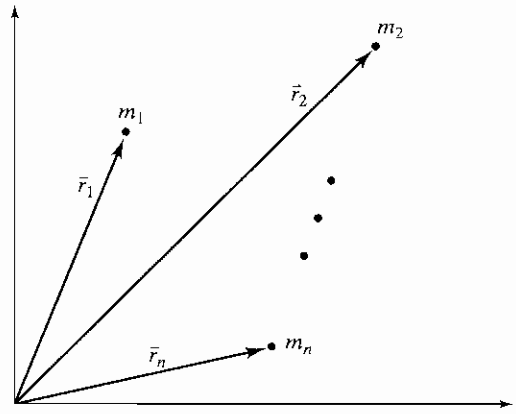
$$\vec{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.$$

a. Why do the first components of $\vec{v}_2$ and $\vec{v}_3$ equal 0?

b. Find the outputs x_1, x_2, x_3 required to satisfy demand.

40. Consider some particles in the plaue with position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ and masses $m_1, m_2, \ldots, m_n$.

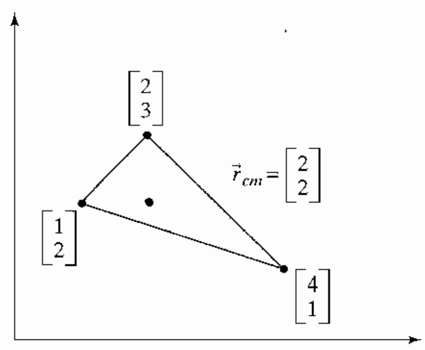


The position vector of the *center of mass* of this system is

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that the plate can be supported at the poiut $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$; that is, $\vec{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? Assume that the mass of the plate itself is negligible.



41. The *momentum* $\vec{P}$ of a system of n particles in space with masses $m_1, m_2, \ldots, m_n$ and velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is defiued as

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n.$$

Now cousider two elementary particles with velocities

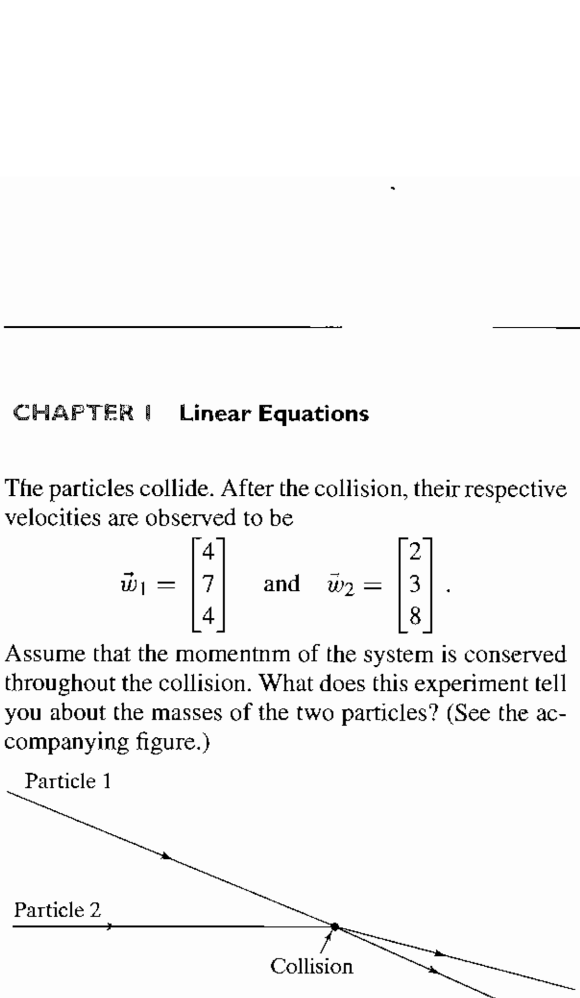
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{aud} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.$$

[11] W. Leontief, *Input–Output Economics*, Oxford University Press, 1966.

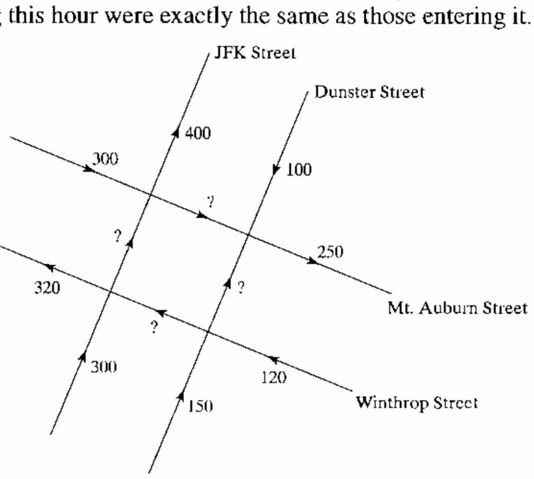
The particles collide. After the collision, their respective velocities are observed to be

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

Assume that the momentnm of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? (See the accompanying figure.)



42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

43. Let $S(t)$ be the length of the tth day of the year 2009 in Mumbai (formerly known as Bombay), India (measured in hours, from suurise to sunset). We are given the following values of $S(t)$:

t	$S(t)$
47	11.5
74	12
273	12

For example, $S(47) = 11.5$ means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function $S(t)$ is well approximated by a trigonometric function of the form

$$S(t) = a + b\cos\left(\frac{2\pi t}{365}\right) + c\sin\left(\frac{2\pi t}{365}\right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

44. Kyle is getting some flowers for Olivia, his Valentine. Being of a precise analytical mind, he plans to spend exactly \$24 on a bunch of exactly two dozen flowers. At the flower market tbey have lilies (\$3 each), roses (\$2 each), and daisies (\$0.50 each). Kyle knows that Olivia loves lilies; what is he to do?

45. Consider the equations

$$\begin{vmatrix} x + 2y + 3z = 4 \\ x + ky + 4z = 6 \\ x + 2y + (k+2)z = 6 \end{vmatrix},$$

where k is an arbitrary constant.

a. For which values of the constaut k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

46. Consider the equations

$$\begin{vmatrix} y + 2kz = 0 \\ x + 2y + 6z = 2 \\ kx + 2z = 1 \end{vmatrix},$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

47. a. Fiud all solutions x_1, x_2, x_3, x_4 of the system $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$.

b. In part (a), is there a solution with $x_1 = 1$ and $x_4 = 13$?

48. For au arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, x_3, \ldots, x_n$ of the simultaueous equations $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$, $\ldots$, $x_{n-1} = \frac{1}{2}(x_{n-2} + x_n)$. Note that we are asked to solve the simultaneous equatious $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$, for $k = 2, 3, \ldots, n-1$.

49. Consider the system

$$\begin{vmatrix} 2x + y = C \\ 3y + z = C \\ x + 4z = C \end{vmatrix},$$

where C is a constant. Find the smallest positive integer C such that x, y, and z are all integers.

50. Find a
$f(0)$ =
you ha
result.

Exercises
is a curve
of the forn
$c_6y^2 = 0$,
nonzero.
parabolas
tions $f(x$
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$-12 + 3x$
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50. Find all the polynomials $f(t)$ of degree ≤ 3 such that $f(0) = 3$, $f(1) = 2$, $f(2) = 0$, and $\int_0^2 f(t)\,dt = 4$. (If you have studied Simpson's rule in calculus, explain the result.)

Exercises 51 through 60 are concerned with conics. A conic is a curve in $\mathbb{R}^2$ that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$, where at least one of the coefficients c_i is nonzero. Examples are circles, ellipses, hyperbolas, and parabolas. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ describe the same conic. For example, the equation $-4 + x^2 + y^2 = 0$ and $-12 + 3x^2 + 3y^2 = 0$ both describe the circle of radius 2 centered at the origin. In Exercises 51 through 60, find all the conics through the given points, and draw a rough sketch of your solution curve(s).

51. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, and $(0, 2)$.

52. $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$, and $(1, 3)$.

53. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(3, 0)$, and $(1, 1)$.

54. $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(1, 0)$.

55. $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$.

56. $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, -1)$.

57. $(5, 0)$, $(1, 2)$, $(2, 1)$, $(8, 1)$, and $(2, 9)$.

58. $(1, 0)$, $(2, 0)$, $(2, 2)$, $(5, 2)$, and $(5, 6)$.

59. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$, and $(1, 1)$.

60. $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$, $(1, 3)$, and $(4, 1)$.

61. Students are buying books for the new semester. Eddie buys the environmeutal statistics book and the set theory book for $178. Leah, who is buying books for herself and her friend, spends $319 on two environmental statistics books, one set theory book, and one educational psychology book. Mehmet buys the educational psychology book and the set theory book for $147 in total. How much does each book cost?

62. Students are buying books for the new semester. Brigitte buys the German grammar book and the German novel, *Die Leiden des jungen Werther*, for €64 in total. Claude spends €98 on the linear algebra text and the German grammar book, while Denise buys the linear algebra text and *Werther*, for €76. How much does each of the three books cost?

63. At the beginning of a political science class at a large university, the srudents were asked which term, *liberal* or *conservative*, best described their political views. They were asked the same question at the end of the course, to see what effect the class discussions had on their views. Of those that characterized themselves as "liberal" initially, 30% held conservative views at the end. Of those who were conservative initially, 40% moved to the liberal camp. It turned out that there were just as many students with conservative views at the end as there had been liberal students at the beginning. Out of the 260 students in the class, how many held liberal and conservative views at the beginning of the course and at the end? (No students joined or dropped the class between the surveys, and they all participated in both surveys.)

64. At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in Section A switch to Section B in the first few weeks of class, while 30% of the students iu Section B switch to A, resulting in a net loss of 4 students for Section B. How large were the two sections at the beginning of the semester? No students dropped Linear Algebra (why would they?) or joined the course late.

Historical Problems

65. Five cows and two sheep together cost ten *liang*[12] of silver. Two cows and five sheep together cost eight *liang* of silver. What is the cost of a cow and a sheep, respectively? (*Nine Chapters*,[13] Chapter 8, Problem 7)

66. If you sell two cows and five sheep and you buy 13 pigs, you gain 1,000 coins. If you sell three cows and three pigs and buy nine sheep, you break eveu. If you sell six sheep and eight pigs and you buy five cows, you lose 600 coins. What is the price of a cow, a sheep, and a pig, respectively? (*Nine Chapters*, Chapter 8, Problem 8)

67. You place five sparrows on one of the pans of a balance and six swallows on the other pan; it turns out that the sparrows are heavier. But if you exchange one sparrow and one swallow, the weights are exactly balanced. All the birds together weigh 1 *jin*. What is the weight of a sparrow and a swallow, respectively? [Give the answer in *liang*, with 1 *jin* = 16 *liang*.] (*Nine Chapters*, Chapter 8, Problem 9)

68. Consider the task of pulling a weight of 40 *dan*[14] np a hill; we have one military horse, two ordinary horses, and three weak horses at our disposal to get the job done. It turns out that the military horse and one of the ordinary horses, pulling together, are barely able to pull the

[12] A *liang* was about 16 grams at the time of the Han Dynasty.

[13] See page 1; we present some of the problems from the *Nine Chapters on the Mathematical Art* in a free translation, with some additional explanations, since the scenarios discussed in a few of these problems are rather unfamiliar to the modern reader.

[14] 1 *dan* = 120 *jin* = 1,920 *liang*. Thus a *dan* was about 30 kilograms at that time.

weight (but they could not pull any more). Likewise, the two ordinary horses together with one weak horse are jnst able to do the job, as are the three weak horses together with the military horse. How much weight can each of the horses pull alone? (*Nine Chapters*, Chapter 8, Problem 12)

69. Five households share a deep well for their water supply. Each household owns a few ropes of a certain length, which varies only from household to household. The five households, A, B, C, D, and E, own 2, 3, 4, 5, and 6 ropes, respectively. Even when tying all their ropes together, none of the households alone are able to reach the water, but A's two ropes together with one of B's ropes just reach the water. Likewise, B's three ropes with one of C's ropes, C's fonr ropes with one of D's ropes, D's five ropes with one of E's ropes, and E's six ropes with one of A's ropes all just reach the water. How long are the ropes of the various households, and how deep is the well?
Commentary: As stated, this problem leads to a system of 5 linear equations in 6 variahles; with the given information, we are unable to determine the depth of the well. The *Nine Chapters* gives one particular solution, where the depth of the well is 7 *zhang*,[15] 2 *chi*, 1 *cun*, or 721 *cun* (since 1 *zhang* = 10 *chi* and 1 *chi* = 10 *cun*). Using this particular value for the depth of the well, find the lengths of the varions ropes.

70. "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?" (From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D.)
Commentary: This famous *Hundred Fowl Problem* has reappeared in conntless variations in Indian, Arabic, and European texts (see Exercises 71 through 74); it has remained popular to this day (see Exercise 44 of this section).

71. "Pigeons are sold at the rate of 5 for 3 panas, sarasabirds at the rate of 7 for 5 panas, swans at the rate of 9 for 7 panas, and peacocks at the rate of 3 for 9 panas. A man was told to hring 100 birds for 100 panas for the amusement of the King's son. What does he pay for each of the various kinds of birds that he buys?" (From the *Ganita-Sara-Sangraha* by Mahavira, India; 9th century A.D.) Find one solution to this problem.

72. "A duck costs four coins, five sparrows cost one coin, and a rooster costs one coin. Somebody buys 100 birds for 100 coins. How many birds of each kind can he buy?" (From the *Key to Arithmetic* by Al-Kashi; 15th century)

73. "A certain person buys sheep, goats, and hogs, to the numher of 100, for 100 crowns; the sheep cost him $\frac{1}{2}$ a crown a-piece; the goats, $1\frac{1}{3}$ crown; and the hogs $3\frac{1}{2}$ crowns. How many had he of each?" (From the *Elements of Algebra* by Leonhard Euler, 1770)

74. "A gentleman has a household of 100 persons and orders that they be given 100 measures of grain. He directs that each man shonld receive three measures, each woman two measures, and each child half a measure. How many men, women, and children are there in this household?" We are told that there is at least one man, one woman, and one child. (From the *Problems for Quickening a Young Mind* by Alcuin [c. 732–804], the Abbot of St. Martins at Tonrs. Alcuin was a friend and tutor to Charlemagne and his family at Aachen.)

75. A father, when dying, gave to his sons 30 barrels, of which 10 were full of wine, 10 were half full, and the last 10 were empty. Divide the wine and flasks so that there will be equal division among the three sons of both wine and barrels. Find all the solutions of this problem. (From Alcuin)

76. "Make me a crown weighing 60 *minae*, mixing gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fonrths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must put in." (From the *Greek Anthology* by Metrodorus, 6th century A.D.)

77. Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together." "Give me the purse and I shall have three times as much as the two of you together" said the second merchant. The third merchant said "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together." If there are 60 coins (of equal value) in the purse, how much money does each merchant have? (From Mahavira)

78. 3 cows graze 1 field hare in 2 days,
7 cows graze 4 fields bare in 4 days. and
3 cows graze 2 fields hare in 5 days.
It is assumed that each field initially provides the same amount, x, of grass; that the daily growth, y, of the fields remains constant; and that all the cows eat the same amount, z, each day. (Qnantities x, y, and z are measured by weight.) Find all the solutions of this problem. (This is a special case of a problem discussed by Isaac Newton in his *Arithmetica Universalis*, 1707.)

[15] 1 *zhang* was about 2.3 meters at that time.

We can generalize:

Theorem 1.3.11 Matrix form of a linear system

We can write the linear system with augmented matrix $\left[A \mid \vec{b}\right]$ in matrix form as

$$A\vec{x} = \vec{b}.$$ ■

Note that the ith component of $A\vec{x}$ is $a_{i1}x_1 + \cdots + a_{im}x_m$, by Definition 1.3.7. Thus, the ith component of the equation $A\vec{x} = \vec{b}$ is

$$a_{i1}x_1 + \cdots + a_{im}x_m = b_i;$$

this is the ith equation of the system with augmented matrix $\left[A \mid \vec{b}\right]$.

EXAMPLE 14 Write the system

$$\left| \begin{matrix} 2x_1 - 3x_2 + 5x_3 = 7 \\ 9x_1 + 4x_2 - 6x_3 = 8 \end{matrix} \right|$$

in matrix form.

Solution

The coefficient matrix is $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$. The matrix form is

$$A\vec{x} = \vec{b}, \quad \text{or,} \quad \begin{bmatrix} 2 & -3 & 5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$ ■

Now that we can write a linear system as a *single equation*, $A\vec{x} = \vec{b}$, rather than a list of simultaneous equations, we can think about it in new ways.

For example, if we have an equation $ax = b$ of *numbers*, we can divide both sides by a to find the solution x:

$$x = \frac{b}{a} = a^{-1}b \quad (\text{if } a \neq 0).$$

It is natural to ask whether we can take an analogous approach in the case of the equation $A\vec{x} = \vec{b}$. Can we "divide by A," in some sense, and write

$$\vec{x} = \frac{\vec{b}}{A} = A^{-1}\vec{b}?$$

This issue of the invertibility of a matrix will be one of the main themes of Chapter 2.

EXERCISES 1.3

GOAL *Use the reduced row-echelon form of the augmented matrix to find the number of solutions of a linear system. Apply the definition of the rank of a matrix. Compute the product $A\vec{x}$ in terms of the rows or the columns of A. Represent a linear system in vector or in matrix form.*

1. The reduced row-echelon forms of the augmented matrices of three systems are given below. How many solutions does each system have?

a. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ **b.** $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Find the rank of the matrices in Exercises 2 through 4.

2. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5. **a.** Write the system

$$\left| \begin{array}{l} x + 2y = 7 \\ 3x + y = 11 \end{array} \right|$$

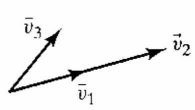
in vector form.

b. Use your answer in part (a) to represent the system geometrically. Solve the system and represent the solution geometrically.

6. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in $\mathbb{R}^2$ (sketched in the accompanying figure). Vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

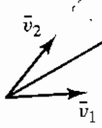
have? Argue geometrically.



7. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ in $\mathbb{R}^2$ shown in the accompanying sketch. How many solutions x, y does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

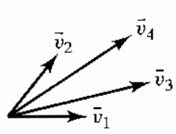
have? Argue geometrically. ⇒ only one parallelogram that produce $\vec{v}_3$ ... $A = [\vec{v}_1\ \vec{v}_2]_{2\times2}$ ⇒ rank(A) = 2 ⇒ ∃ unique solution $\binom{x}{y}$ in $A\binom{x}{y} = \vec{v}_3$



8. Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ in $\mathbb{R}^2$ shown in the accompanying sketch. Arguing geometrically, find two solutions x, y, z of the linear system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{v}_4.$$

How do you know that this system has in fact infinitely many solutions?



9. Write the system

$$\left| \begin{array}{l} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 4 \\ 7x + 8y + 9z = 9 \end{array} \right|$$

in matrix form.

Compute the dot products in Exercises 10 through 12 (if the products are defined).

10. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 13 through 15 using paper and pencil. In each case, compute the product two ways: in terms of the columns of A (Theorem 1.3.8) and in terms of the rows of A (Definition 1.3.7).

13. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

Compute the products $A\vec{x}$ in Exercises 16 through 19 using paper and pencil (if the products are defined).

16. $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & -1 \\ -5 & 1 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

20. **a.** Find $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}$.

b. Find $9 \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

21. Use technology to compute the product

$$\begin{bmatrix} 1 & 7 & 8 & 9 \\ 1 & 2 & 9 & 1 \\ 1 & 5 & 1 & 5 \\ 1 & 6 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 9 \\ 5 \\ 6 \end{bmatrix}.$$

22. Consider a linear system of three equations with three unknowns. We are told that the system has a uuique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

23. Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

24. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

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25. Let A be a 4×4 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

26. Let A be a 4×3 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the number of solutions of the system $A\vec{x} = \vec{c}$?

27. If the rank of a 4×4 matrix A is 4, what is $\text{rref}(A)$?

28. If the rank of a 5×3 matrix A is 3, what is $\text{rref}(A)$?

In Problems 29 through 32, let $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -9 \end{bmatrix}$ ***and*** $\vec{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

29. Find a *diagonal* matrix A such that $A\vec{x} = \vec{y}$.

30. Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.

31. Find an *upper triangular* matrix A such that $A\vec{x} = \vec{y}$. Also, it is required that all the entries of A on and above the diagonal be nonzero.

32. Find a matrix A with all nonzero entries such that $A\vec{x} = \vec{y}$.

33. Let A be the $n \times n$ matrix with all 1's on the diagonal and all 0's above and below the diagonal. What is $A\vec{x}$, where $\vec{x}$ is a vector in $\mathbb{R}^n$?

34. We define the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$.

a. For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix},$$

compute $A\vec{e}_1$, $A\vec{e}_2$, and $A\vec{e}_3$.

b. If B is an $n \times 3$ matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, what is $B\vec{e}_1$, $B\vec{e}_2$, $B\vec{e}_3$?

35. In $\mathbb{R}^m$, we define

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th component.}$$

If A is an $n \times m$ matrix, what is $A\vec{e}_i$?

36. Find a 3×3 matrix A such that

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

$$\text{and} \quad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

37. Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

38. **a.** Using technology, generate a random 3×3 matrix A. (The entries may be either single-digit integers or numbers between 0 and 1, depending on the technology you are using.) Find $\text{rref}(A)$. Repeat this experiment a few times.

b. What does the reduced row-echelon form of most 3×3 matrices look like? Explain.

39. Repeat Exercise 38 for 3×4 matrices.

40. Repeat Exercise 38 for 4×3 matrices.

41. How many solutions do most systems of three linear equations with three unknowns have? Explain in terms of your work in Exercise 38.

42. How many solutions do most systems of three linear equations with four unknowns have? Explaiu in terms of your work in Exercise 39.

43. How many solutions do most systems of four linear equations with three unknowns have? Explain in terms of your work in Exercise 40.

44. Consider an $n \times m$ matrix A with more rows than columns ($n > m$). Show that there is a vector $\vec{b}$ in $\mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ is inconsistent.

45. Consider an $n \times m$ matrix A, a vector $\vec{x}$ in $\mathbb{R}^m$, and a scalar k. Show that

$$A(k\vec{x}) = k(A\vec{x}).$$

46. Find the rank of the matrix

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$

where a, d, and f are nonzero, and b, c, and e are arbitrary numbers.

47. A linear system of the form

$$A\vec{x} = \vec{0}$$

is called *homogeneous*. Justify the following facts:

a. All homogeneous systems are consistent.

b. A homogeneous system with fewer equations than unknowns has infinitely many solutions.

c. If $\vec{x}_1$ and $\vec{x}_2$ are solutions of the homogeneous system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_2$ is a solution as well.

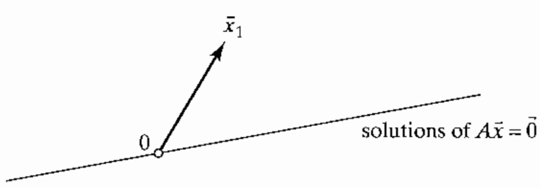
d. If $\vec{x}$ is a solution of the homogeneous system $A\vec{x} = \vec{0}$ and k is an arbitrary constant, then $k\vec{x}$ is a solution as well.

48. Consider a solution $\vec{x}_1$ of the linear system $A\vec{x} = \vec{b}$. Justify the facts stated in parts (a) and (b):

a. If $\vec{x}_h$ is a solution of the system $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution of the system $A\vec{x} = \vec{b}$.

b. If $\vec{x}_2$ is another solution of the system $A\vec{x} = \vec{b}$, then $\vec{x}_2 - \vec{x}_1$ is a solution of the system $A\vec{x} = \vec{0}$.

c. Now suppose A is a 2×2 matrix. A solution vector $\vec{x}_1$ of the system $A\vec{x} = \vec{b}$ is shown in the accompanying figure. We are told that the solutions of the system $A\vec{x} = \vec{0}$ form the line shown in the sketch. Draw the line consisting of all solutions of the system $A\vec{x} = \vec{b}$.



If you are puzzled by the generality of this problem, think about an example first:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

49. Consider the accompanying table. For some linear systems $A\vec{x} = \vec{b}$, you are given either the rank of the coefficient matrix A, or the rank of the augmented matrix $\left[A \mid \vec{b}\right]$. In each case, state whether the system could have no solution, one solution, or infinitely many solutions. There may be more than one possibility for some systems. Justify yonr answers.

	Number of Equations	Number of Unknowns	Rank of A	Rank of $\left[A \mid \vec{b}\right]$
a.	3	4	—	2
b.	4	3	3	—
c.	4	3	—	4
d.	3	4	3	—

50. Consider a linear system $A\vec{x} = \vec{b}$, where A is a 4×3 matrix. We are told that rank $\left[A \mid \vec{b}\right] = 4$. How many solutions does this system have?

51. Consider an $n \times m$ matrix A, an $r \times s$ matrix B, and a vector $\vec{x}$ in $\mathbb{R}^p$. For which values of n, m, r, s, and p is the product

$$A(B\vec{x})$$

defined?

52. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Can you find a 2×2 matrix C such that

$$A(B\vec{x}) = C\vec{x},$$

for all vectors $\vec{x}$ in $\mathbb{R}^2$?

53. If A and B are two $n \times m$ matrices, is

$$(A + B)\vec{x} = A\vec{x} + B\vec{x}$$

for all $\vec{x}$ in $\mathbb{R}^m$?

54. Consider two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ that are not parallel. Which vectors in $\mathbb{R}^3$ are linear combinations of $\vec{v}_1$ and $\vec{v}_2$? Describe the set of these vectors geometrically. Include a sketch in your answer.

55. Is the vector $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ a linear combination of

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}?$$

56. Is the vector

$$\begin{bmatrix} 30 \\ -1 \\ 38 \\ 56 \\ 62 \end{bmatrix}$$

a linear combination of

$$\begin{bmatrix} 1 \\ 7 \\ 1 \\ 9 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -5 \\ 4 \\ 7 \\ 9 \end{bmatrix}?$$

57. Expres

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58. For w

$\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$

59. For w

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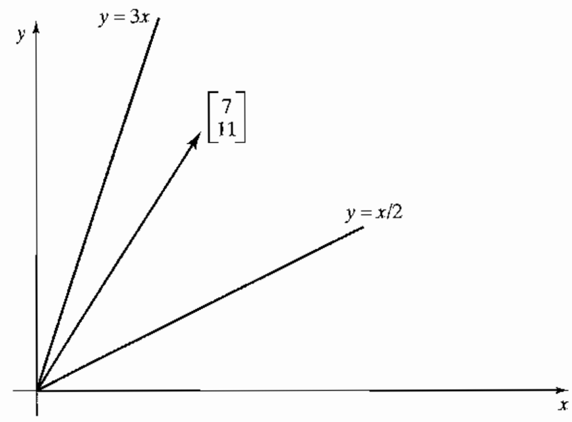
61. For

com

57. Express the vector $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ as the sum of a vector on the line $y = 3x$ and a vector ou the line $y = x/2$.



58. For which values of the constants b and c is the vector $\begin{bmatrix} 3 \\ b \\ c \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$?

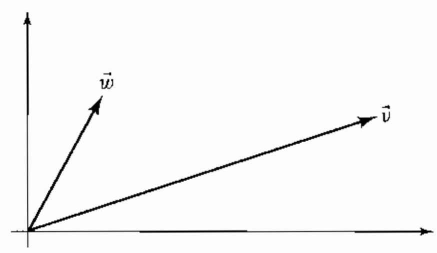
59. For which values of the constants c and d is $\begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$?

60. For which values of the constants a, b, c, and d is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}$?

61. For which values of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$?

62. For which valnes of the constant c is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \\ b^2 \end{bmatrix}$, where a and b are arbitrary constants?

In Exercises 63 through 67, consider the vectors $\vec{v}$ and $\vec{w}$ in the accompanying figure.



63. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where c is an arbitrary real nnmber.

64. Give a geometrical description of the set of all vectors of the form $\vec{v} + c\vec{w}$, where $0 \le c \le 1$.

65. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \le a \le 1$ and $0 \le b \le 1$.

66. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $a + b = 1$.

67. Give a geometrical description of the set of all vectors of the form $a\vec{v} + b\vec{w}$, where $0 \le a$, $0 \le b$, and $a + b \le 1$.

68. Give a geometrical description of the set of all vectors $\vec{u}$ in $\mathbb{R}^2$ such that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$.

69. Solve the linear system

$$\left| \begin{array}{rcrcl} & & y & + z & = a \\ x & & & + z & = b \\ x & + & y & & = c \end{array} \right|,$$

where a, b, and c are arbitrary constants.

70. Let A be the $n \times n$ matrix with 0's on the main diagonal, and 1's everywhere else. For an arbitrary vector $\vec{b}$ in $\mathbb{R}^n$, solve the linear system $A\vec{x} = \vec{b}$, expressing the components $x_1, \ldots, x_n$ of $\vec{x}$ in terms of the components of $\vec{b}$. (See Exercise 69 for the case $n = 3$.)

EXERCISES 2.1

GOAL *Use the concept of a linear transformation in terms of the formula $\vec{y} = A\vec{x}$, and interpret simple linear transformations geometrically. Find the inverse of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ (if it exists). Find the matrix of a linear transformation column by column.*

Consider the transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined in Exercises 1 through 3. Which of these transformations are linear?

1. $y_1 = 2x_2$, $y_2 = x_2 + 2$, $y_3 = 2x_2$

2. $y_1 = 2x_2$, $y_2 = 3x_3$, $y_3 = x_1$

3. $y_1 = x_2 - x_3$, $y_2 = x_1 x_3$, $y_3 = x_1 - x_2$

4. Find the matrix of the linear transformation

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3. \end{aligned}$$

5. Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}7\\11\end{bmatrix}, \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}6\\9\end{bmatrix},$$

$$\text{and} \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-13\\17\end{bmatrix}.$$

Find the matrix A of T.

6. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

Is this transformation linear? If so, find its matrix.

7. Snppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$.

8. Find the inverse of the linear transformation

$$\begin{aligned} y_1 &= x_1 + 7x_2 \\ y_2 &= 3x_1 + 20x_2. \end{aligned}$$

In Exercises 9 through 12, decide whether the given matrix is invertible. Find the inverse if it exists. In Exercise 12, the constant k is arbitrary.

9. $\begin{bmatrix}2&3\\6&9\end{bmatrix}$ **10.** $\begin{bmatrix}1&2\\4&9\end{bmatrix}$

11. $\begin{bmatrix}1&2\\3&9\end{bmatrix}$ **12.** $\begin{bmatrix}1&k\\0&1\end{bmatrix}$

13. Prove the following facts:

a. The 2×2 matrix

$$A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$. (*Hint*: Consider the cases $a \neq 0$ and $a = 0$ separately.)

b. If

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

is invertible, then

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$$

[The formula in part (b) is worth memorizing.]

14. a. For which valnes of the constant k is the matrix $\begin{bmatrix}2&3\\5&k\end{bmatrix}$ invertible?

b. For which values of the constant k are all entries of $\begin{bmatrix}2&3\\5&k\end{bmatrix}^{-1}$ integers? (See Exercise 13.)

15. For which values of the constants a and b is the matrix

$$A = \begin{bmatrix}a&-b\\b&a\end{bmatrix}$$

invertible? What is the inverse in this case? (See Exercise 13.)

Give a geometric interpretation of the linear transformations defined by the matrices in Exercises 16 through 23. Show the effect of these transformations on the letter L *considered in Example 5. In each case, decide whether the transformation is invertible. Find the inverse if it exists, and interpret it geometrically. See Exercise 13.*

16. $\begin{bmatrix}3&0\\0&3\end{bmatrix}$ **17.** $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ **18.** $\begin{bmatrix}0.5&0\\0&0.5\end{bmatrix}$

19. $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ **20.** $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ **21.** $\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

22. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ **23.** $\begin{bmatrix}0&2\\-2&0\end{bmatrix}$

Consider the circular face in the accompanying figure. For each of the matrices A in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ on this face.

24. $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

27. $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

30. $\begin{bmatrix}0&0\\0&1\end{bmatrix}$

31. In Chapt… shows th… transform… back?

32. Find an … in $\mathbb{R}^n$.

33. Conside… any vect… wise dir…

You are… be show…

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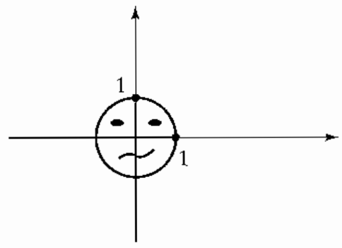
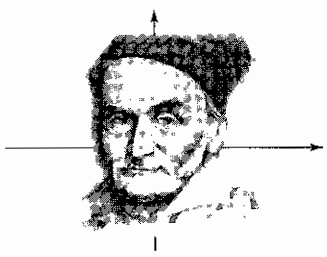


24. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **25.** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ **26.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **28.** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **29.** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

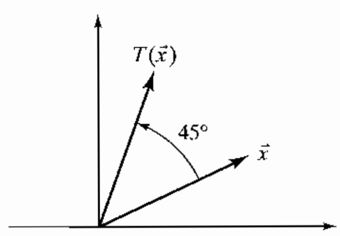
30. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

31. In Chapter 1, we mentioned that an old German bill shows the mirror image of Gauss's likeness. What linear transformation T can you apply to get the actual picture back?



32. Find an $n \times n$ matrix A such that $A\vec{x} = 3\vec{x}$, for all $\vec{x}$ in $\mathbb{R}^n$.

33. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through an angle of 45° in the counterclockwise direction, as shown in the following figure:



You are told that T is a linear transformation. (This will be shown in the next section.) Find the matrix of T.

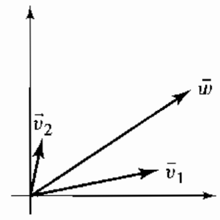
34. Consider the transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through a given angle θ in the counterclockwise direction. (Compare this with Exercise 33.) You are told that T is linear. Find the matrix of T in terms of θ.

35. In the example about the French coast guard in this section, suppose you are a spy watching the boat and listening in on the radio messages from the boat. You collect the following data:

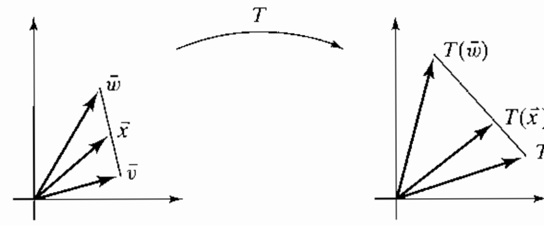
- When the actual position is $\begin{bmatrix} 5 \\ 42 \end{bmatrix}$, they radio $\begin{bmatrix} 89 \\ 52 \end{bmatrix}$.
- When the actual position is $\begin{bmatrix} 6 \\ 41 \end{bmatrix}$, they radio $\begin{bmatrix} 88 \\ 53 \end{bmatrix}$.

Can you crack their code (i.e., find the coding matrix), assuming that the code is linear?

36. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{w}$ be three vectors in $\mathbb{R}^2$, as shown below. We are told that $T(\vec{v}_1) = \vec{v}_1$ and $T(\vec{v}_2) = 3\vec{v}_2$. On the same axes, sketch $T(\vec{w})$.



37. Consider a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose that $\vec{v}$ and $\vec{w}$ are two arbitrary vectors in $\mathbb{R}^2$ and that $\vec{x}$ is a third vector whose endpoint is on the line segment connecting the endpoints of $\vec{v}$ and $\vec{w}$. Is the endpoint of the vector $T(\vec{x})$ necessarily on the line segment connecting the endpoints of $T(\vec{v})$ and $T(\vec{w})$? Justify yonr answer.

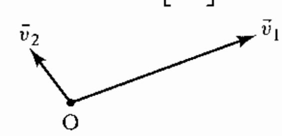


[*Hint*: We can write $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, for some scalar k between 0 and 1.]

We can summarize this exercise by saying that a linear transformation maps a line onto a line.

38. The two column vectors $\vec{v}_1$ and $\vec{v}_2$ of a 2×2 matrix A are shown in the accompanying sketch. Consider the linear transformation $T(\vec{x}) = A\vec{x}$, from $\mathbb{R}^2$ to $\mathbb{R}^2$. Sketch the vector

$$T\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$



39. Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m),$$

where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ are the standard vectors in $\mathbb{R}^m$.

40. Describe all linear transformations from $\mathbb{R}$ $(= \mathbb{R}^1)$ to $\mathbb{R}$. What do their graphs look like?

41. Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$ $(= \mathbb{R}^1)$. What do their graphs look like?

42. When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a compnter screen), you have to transform spatial coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

into plane coordinates, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The simplest choice is a linear transformation, for example, the one given by the matrix

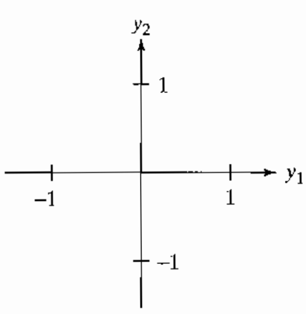
$$\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Include the images of the x_1, x_2, and x_3 axes in your sketch:



b. Represent the image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ in your figure in part (a). Explain.

c. Find all the points

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ in } \mathbb{R}^3$$

that are transformed to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain.

43. a. Consider the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ (the dot product) from $\mathbb{R}^3$ to $\mathbb{R}$ linear? If so, find the matrix of T.

b. Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$ linear? If so, find the matrix of T (in terms of the components of $\vec{v}$).

c. Conversely, consider a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}$. Show that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{v} \cdot \vec{x}$, for all $\vec{x}$ in $\mathbb{R}^3$.

44. The cross prodnct of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

(See Definition A.9 and Theorem A.11 in the Appendix.) Consider an arbitrary vector $\vec{v}$ in $\mathbb{R}^3$. Is the transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\vec{v}$.

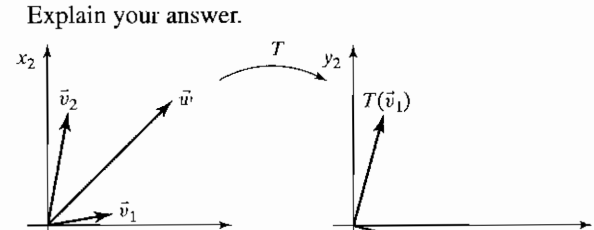
45. Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$, where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$ and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L(T(\vec{x}))$ linear as well? [The transformation $\vec{z} = L(T(\vec{x}))$ is called the *composite* of T and L.]

46. Let

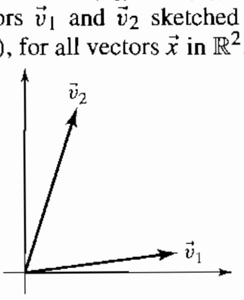
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Find the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$. (See Exercise 45.) [*Hint:* Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.]

47. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Three vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{w}$ in $\mathbb{R}^2$ and the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$ are shown in the accompanying figure. Sketch $T(\vec{w})$. Explain your answer.



48. Consider two linear transformations T and L from $\mathbb{R}^2$ to $\mathbb{R}^2$. We are told that $T(\vec{v}_1) = L(\vec{v}_1)$ and $T(\vec{v}_2) = L(\vec{v}_2)$ for the vectors $\vec{v}_1$ and $\vec{v}_2$ sketched below. Show that $T(\vec{x}) = L(\vec{x})$, for all vectors $\vec{x}$ in $\mathbb{R}^2$.



49. Some p accept
a. A p How
b. Fin
into
c. Is t inv ans

50. A gol make 20 and
a. Kir gol 5,0 all gol fri tak the "E pa fin mu Is
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c. Is ve ar

51. The c to Ce the s niqu conv
a. F
b

49. Some parking meters in downtown Geneva, Switzerland, accept 2 Franc and 5 Franc coins.

a. A parking officer collects 51 coins worth 144 Francs. How many coins are there of each kind?

b. Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{number of 2 Franc coins} \\ \text{number of 5 Franc coins} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix}.$$

c. Is the matrix A in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

50. A goldsmith uses a platinum alloy and a silver alloy to make jewelry; the densities of these alloys are exactly 20 and 10 grams per cubic centimeter, respectively.

a. King Hiero of Syracuse orders a crown from this goldsmith, with a total mass of 5 kilograms (or 5,000 grams), with the stipulation that the platinum alloy must make up at least 90% of the mass. The goldsmith delivers a beautiful piece, but the king's friend Archimedes has doubts about its purity. While taking a bath, he comes up with a method to check the composition of the crown (famously shouting "Eureka!" in the process, and running to the king's palace naked). Submerging the crown in water, he finds its volume to be 370 cubic centimeters. How much of each alloy went into this piece (by mass)? Is this goldsmith a crook?

b. Find the matrix A that transforms the vector

$$\begin{bmatrix} \text{mass of platinum alloy} \\ \text{mass of silver alloy} \end{bmatrix}$$

into the vector

$$\begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix},$$

for any piece of jewelry this goldsmith makes.

c. Is the matrix A in part (b) invertible? If so, find the inverse (use Exercise 13). Use the result to check your answer in part (a).

51. The conversion formula $C = \frac{5}{9}(F-32)$ from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.

a. Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$. (The second row of A will be $\begin{bmatrix} 0 & 1 \end{bmatrix}$.)

b. Is the matrix A in part (a) invertible? If so, find the inverse (use Exercise 13). Use the result to write a formula expressing F in terms of C.

52. In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (C\$) and the South African Rand (ZAR). Consider the matrix

$$A = \begin{array}{c} \begin{array}{cc} \text{C\$} & \text{ZAR} \end{array} \\ \begin{bmatrix} 1 & 1/8 \\ 8 & 1 \end{bmatrix} \end{array} \begin{array}{l} \text{C\$} \\ \text{ZAR} \end{array}$$

representing the fact that C\$1 is worth ZAR8 (as of June 2008).

a. After a trip you have C\$100 and ZAR1,600 in your pocket. We represent these two values in the vector $\vec{x} = \begin{bmatrix} 100 \\ 1{,}600 \end{bmatrix}$. Compute $A\vec{x}$. What is the practical significance of the two components of the vector $A\vec{x}$?

b. Verify that matrix A fails to be invertible. For which vectors $\vec{b}$ is the system $A\vec{x} = \vec{b}$ consistent? What is the practical significance of your answer? If the system $A\vec{x} = \vec{b}$ is consistent, how many solutions $\vec{x}$ are there? Again, what is the practical significance of the answer?

53. Consider a larger currency exchange matrix (see Exercise 52), involving four of the world's leading currencies: Euro (€), U.S. dollar (\$), Japanese yen (¥), and British pound (£).

$$A = \begin{array}{c} \begin{array}{cccc} € & \$ & ¥ & £ \end{array} \\ \begin{bmatrix} * & \frac{5}{8} & * & * \\ * & * & * & 2 \\ 170 & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} \begin{array}{l} € \\ \$ \\ ¥ \\ £ \end{array}$$

The entry a_{ij} gives the value of one unit of the jth currency, expressed in terms of the ith currency. For example, $a_{31} = 170$ means that €1 = ¥170 (as of June 2008). Find the exact values of the 13 missing entries of A (expressed as fractions).

54. Consider an arbitrary currency exchange matrix A (see Exercises 52 and 53).

a. What are the diagonal entries a_{ii} of A?

b. What is the relationship between a_{ij} and a_{ji}?

c. What is the relationship between a_{ik}, a_{kj}, and a_{ij}?

d. What is the rank of A? What is the relationship between A and $\text{rref}(A)$?