DIMENSION INVARIANCE OF FINITE FRAMES OF TRANSLATES AND GABOR FRAMES

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Abstract. A dimension invariance property of finite frames of translates and Gabor frames is discussed. Under appropriate support conditions among the frame and dual frame generating functions, we show that a pair of dual frames evaluated in a given space remains a valid dual set if they are naturally embedded in the underlying space of almost arbitrarily enlarged dimension. Consequently, the evaluation of duals in a very large dimensional space is now easily accessible by merely working in a space of some much smaller dimension. A number of uniform and non-uniform schemes are studied. To satisfy the support conditions, a method of finding compactly supported alternate dual functions via a known parametric dual frame formula is discussed. Oftentimes it is convenient to have truncated approximate duals that satisfy the support conditions. Stability studies of the dimension invariance principle via such approximate duals are also presented.

1. Introduction

In real world implementations, finite frames of very large dimension can present great difficulties in terms of computational feasibilities. Take super-resolution image fusion application for example, suppose we are to fusion 4 images of the size 256×256 into one high resolution image, by any fusion algorithm, including the frame fundamental image fusion principle [26], there will be a composite frame matrix M of the size $2^{18} \times 2^{18}$. To find a dual frame of such a composite frame system M, direct computation in the non-separable way is highly challenging if not impossible even by the state-of-the-art computing technology. Other numerical means maybe possible to facilitate the computation. Dimension invariance property can be one that is indispensable.

In this article we will present the dimension invariance property of a certain family of frames of translates for finite dimensional Hilbert spaces. This dimension reduction principle reduces the dual frame evaluation to a much smaller dimensional space while extending the solution to an arbitrarily large dimension through natural embeddings.

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Given a separate Hilbert space $\mathcal{H}$, a subset $\{g_k\} \subseteq \mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exists positive constants $A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_k |\langle f, g_k\rangle|^2 \leq B||f||^2$$

for every $f \in \mathcal{H}$. Given a frame $\{g_k\}$, the invertible *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$S(f) = \sum_k \langle f, g_k\rangle g_k.$$

Standard frame representation refers to

$$\forall f \in \mathcal{H}, \qquad f = \sum_j \langle f, g_k\rangle S^{-1} g_k = \sum_j \langle f, S^{-1} g_k\rangle g_k,$$

where $\{S^{-1}g_k\}$ is the *canonical dual frame* to $\{g_k\}$. More generally, any frame $\{g_k^*\}$ satisfying

$$f = \sum_j \langle f, g_k\rangle g_k^* = \sum_k \langle f, g_k^*\rangle g_k$$

for all $f \in \mathcal{H}$, is said to be an *alternative dual frame* to $\{g_k\}$. In general, a given frame may have infinitely many alternative dual frames. If a sequence of vectors $\{g_k\}$ forms a frame of its own (closed) linear span, it is often referred as a *frame sequence* (without further specifying the subspace span$\{g_k\}$). Mathematical fundamentals of frames can be found in, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [12], [11], [13], [14], [15], [16], [18] and [19].

Let $\mathcal{H} = \mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and think of a vector $f \in \mathcal{H}$ as a function $f : \mathbb{Z}_n \to \mathbb{F}$, where $\mathbb{Z}_n$ is the standard notation for the set of natural numbers 0 to $n-1$. Define the *translation operator* $T_c : \mathcal{H} \to \mathcal{H}$ (with periodic treatment at boundaries) by the formula

$$T_c f(x) = f(x - c)$$

where $c \in \mathbb{Z}_n$. A *multi-frame of translates* is a frame of the form $\{T_{kc_j} g^{(j)}\}_{j,k}$, where $1 \leq j \leq \ell$, and ℓ is a finite number. For each j, c_j is a divisor of n, and k ranges from 0 to $\frac{n}{c_j}$. For simplicity, we shall term $\{T_{kc_j} g^{(j)}\}_{j,k}$ simply a *frame of translates*. The multiplicity will speak for itself with multiple wave functions $\{g^{(j)}\}_{j=1}^{\ell}$.

Frames of translates and Gabor frame have been extremely useful in a variety of signal processing applications. In this article we derive conditions under which a dual pair of frames of translates (and Gabor frames) evaluated in a small dimension n will remain valid as the dimension $\tilde{n}$ of the underlying space becomes arbitrarily large, provided all c_j divide $\tilde{n}$. This drastically reduces computational complexity and makes frame application feasible in super large signal processing (such as image) applications.

The paper is organized as follows, in *Section 2* we will state and prove the basic dimension invariance condition for frames of uniform translates in terms of the combined support of waveforms $\{g^{(j)}\}_{j=1}^{\ell}$ and its duals $\{h^{(j)}\}_{j=1}^{\ell}$. In *Sections 3, 4*, and *5*, respectively, we will derive dimension invariance conditions for frames of non-uniform but proportional translates, for frame of translates in multi-variables, and for Gabor frames and multi-Gabor frames with uniform and non-uniform parameters. In *Section 6* we show that the support condition in our basic result can be achieved through alternate dual evaluation via a parametric dual frame formula [21], and finally in *Section 7* we prove the validity of the dimension invariance principle while truncated approximate duals are used to satisfy the dimension invariance (support) condition. Error estimates are derived via such approximate duals and the dimension invariance principle.

2. Dimension invariance of frames of uniform translates

Let $\mathcal{H} = \mathbb{F}^n$ be as before. For a vector $f \in \mathcal{H}$, we define $\text{supp}(f)$ to be the smallest domain that contains all nonzero components of f, and $|\text{supp}(f)|$ the measure/length of $\text{supp}(f)$.

We will be interested in pairs of frames of (uniform) translates of the form $\{T_{kc}g^{(j)}\}_{j,k}$ and $\{T_{kc}h^{(j)}\}_{j,k}$ which satisfy, for all $f \in \mathcal{X} \equiv \text{span}\{T_{kc}g^{(j)}\} \subseteq \mathcal{H}$,

$$f = \sum_{j=1}^{\ell} \sum_{k=0}^{n/c-1} \langle f, T_{kc}h^{(j)} \rangle T_{kc}g^{(j)} \tag{1}$$

i.e., $\{T_{kc}g^{(j)}\}$ and $\{T_{kc}h^{(j)}\}$ form a dual pair of frame sequences.

Notation of natural embedding. Given two Hilbert spaces $\mathcal{H} = \mathbb{F}^n$ and $\tilde{\mathcal{H}} = \mathbb{F}^{\tilde{n}}$ such that $n \leq \tilde{n}$ let $\{e_k\}_{k=0}^{n-1}$ be the standard orthonormal basis of $\mathcal{H}$ and let $\{d_k\}_{k=0}^{\tilde{n}-1}$ be the standard orthonormal basis for $\tilde{\mathcal{H}}$ and let $\iota : \mathcal{H} \to \tilde{\mathcal{H}}$ be given by $\iota(e_k) = d_k$ $k = 0, 1, ..., n-1$. We will use the notation $\tilde{f} = \iota(f)$ for $f \in \mathcal{H}$. (We should think of $\tilde{f}$ as the natural embedding of f in $\tilde{\mathcal{H}}$ by padding with zeros.)

Theorem 2.1. *Let $\mathcal{H} = \mathbb{F}^n$. Suppose $\{T_{kc}g^{(j)}\}$ and $\{T_{kc}h^{(j)}\}$ are a dual pair of frames of translates for $\mathcal{X} = span\{T_{kc}g^{(j)}\} \subseteq \mathcal{H}$ which also satisfy*

$$\max_j \left| supp(g^{(j)}) \right| + \max_j \left| supp(h^{(j)}) \right| \leq n. \tag{2}$$

Let $\tilde{\mathcal{H}} = \mathbb{F}^{\tilde{n}}$ where c divides $\tilde{n}$, then $\{T_{kc}\tilde{g}^{(j)}\}$ and $\{T_{kc}\tilde{h}^{(j)}\}$ remain a dual pair of uniform frames of translates for $\tilde{\mathcal{X}} = span\{T_{kc}\tilde{g}^{(j)}\} \subseteq \tilde{\mathcal{H}}$. Here $\tilde{\theta} \in \tilde{\mathcal{H}}$ stands for the natural embedding of $\theta \in \mathcal{H}$.

Proof. Let $N = \frac{n}{c}$ and $\tilde{N} = \frac{\tilde{n}}{c}$. Note that our assumption (2) implies that there exists $0 < N_0 \leq N_1 \leq N-1$ such that $\langle g^{(j)}, T_{kc}h^{(i)} \rangle = 0$ for all $k = N_0, N_0+1, ..., N_1 - 1$ and all i and j. Thus we have the equation

$$\begin{aligned} g^{(j)} &= \sum_{i=1}^{\ell} \sum_{k=0}^{N-1} \langle g^{(j)}, T_{kc}h^{(i)} \rangle T_{kc}g^{(i)} \\ &= \sum_{i=1}^{\ell} \sum_{k=0}^{N_0-1} \langle g^{(j)}, T_{kc}h^{(i)} \rangle T_{kc}g^{(i)} + \sum_{i=1}^{\ell} \sum_{k=N_1}^{N-1} \langle g^{(j)}, T_{kc}h^{(i)} \rangle T_{kc}g^{(i)} \end{aligned}$$

Also observe that

$$T_{kc}\tilde{h}^{(j)}(x) = T_{kc}h^{(j)}(x), \qquad k \in \mathbb{Z}_{N_0}, x \in \mathbb{Z}_n, j \in \{1, ..., \ell\}$$

which implies that

$$\langle \tilde{g}^{(j)}, T_{kc}\tilde{h}^{(i)} \rangle = \langle g^{(j)}, T_{kc}h^{(i)} \rangle, \qquad k \in \mathbb{Z}_{N_0}, i \in \{1, ..., \ell\},$$

and

$$T_{(\tilde{N}-N+k)c}\tilde{h}^{(j)}(x) = T_{kc}h^{(j)}(x), \qquad k \in \mathbb{Z}_N \setminus \mathbb{Z}_{N_1}, x \in \mathbb{Z}_{cN_0}, j \in \{1, ..., \ell\}$$

and

$$T_{(\tilde{N}-N+k)c}\tilde{h}^{(j)}(-x) = T_{kc}h^{(j)}(-x), \qquad k \in \mathbb{Z}_N \setminus \mathbb{Z}_{N_1}, x \in \mathbb{Z}_{c(N-N_1)}, j \in \{1, ..., \ell\}$$

which together imply that

$$\langle \tilde{g}^{(j)}, T_{(\tilde{N}-N+k)c}\tilde{h}^{(i)} \rangle = \langle g^{(j)}, T_{kc}h^{(i)} \rangle, \qquad k \in \mathbb{Z}_N \setminus \mathbb{Z}_{N_1}, i \in \{1, ..., \ell\}.$$

We also have

$$\langle \tilde{g}^{(j)}, T_{kc}\tilde{h}^{(i)} \rangle = 0, \qquad k \in \mathbb{Z}_{\tilde{N}-N+N_1} \setminus \mathbb{Z}_{N_0}.$$

Together these relationships imply

$$\tilde{g}^{(j)} = \sum_{i=1}^{\ell} \sum_{k=0}^{\tilde{N}-1} \langle \tilde{g}^{(j)}, T_{kc}\tilde{h}^{(i)} \rangle T_{kc}\tilde{g}^{(i)}. \tag{3}$$

Now apply the translation operator T_{pc} to (3) to see

$$\begin{aligned}
T_{pc}\tilde{g}^{(j)} &= \sum_{i=1}^{\ell} \sum_{k=0}^{\tilde{N}-1} \langle \tilde{g}^{(j)}, T_{kc}\tilde{h}^{(i)} \rangle T_{pc}T_{kc}\tilde{g}^{(i)} \\
&= \sum_{i=1}^{\ell} \sum_{k=0}^{\tilde{N}-1} \langle \tilde{g}^{(j)}, T_{kc}\tilde{h}^{(i)} \rangle T_{(p+k)c}\tilde{g}^{(i)} \\
&= \sum_{i=1}^{\ell} \sum_{m=p}^{\tilde{N}+p-1} \langle \tilde{g}^{(j)}, T_{(m-p)c}\tilde{h}^{(i)} \rangle T_{mc}\tilde{g}^{(i)} \\
&= \sum_{i=1}^{\ell} \sum_{m=p}^{\tilde{N}+p-1} \langle T_{pc}\tilde{g}^{(j)}, T_{mc}\tilde{h}^{(i)} \rangle T_{mc}\tilde{g}^{(i)} \\
&= \sum_{i=1}^{\ell} \sum_{m=0}^{\tilde{N}-1} \langle T_{pc}\tilde{g}^{(j)}, T_{mc}\tilde{h}^{(i)} \rangle T_{mc}\tilde{g}^{(i)}.
\end{aligned}$$

□

For an earlier version of this result in the context of Gabor frames see [20].

Remark: Note that if $\text{span}\{T_{kc}g^{(j)}\} = \mathcal{H}$ then we also have $\text{span}\{T_{kc}\tilde{g}^{(j)}\} = \tilde{\mathcal{H}}$. To see this just let the standard basis vectors be $\{e_k\}_{k=0}^{n-1}$ for $\mathcal{H}$ and $\{d_k\}_{k=0}^{\tilde{n}-1}$ play the role of the $g^{(j)}$'s in the proof of the theorem.

Corollary 2.2. *Let* $\{T_{kc}g^{(j)}\}$ *be a Parseval frame of translates of span*$\{T_{kc}g^{(j)}\} \subseteq \mathbb{F}^n$ *such that* $\max_j$ *supp*$(g^{(j)}) \leq \frac{n}{2}$. *Then* $\{T_{kc}\tilde{g}^{(j)}\}$ *is a Parseval frame of translates for span*$\{T_{kc}\tilde{g}^{(j)}\} \subseteq \mathbb{F}^{\tilde{n}}$, *where* $\tilde{n}$ *is a multiple of* c.

3. Proportional, non-uniform frames of translates

Notice that in Theorem 2.1 we assumed that all of the vectors $g^{(j)}$ have the same translation parameter c. At this time it is unknown if an analog of Theorem 2.1 exists when each vector $g^{(j)}$ has an arbitrary shift parameter c_j; however, there is one special case worth mentioning when the translation parameters c_j are proportional.

Suppose that we have a dual pair $\{T_{kc_j}g^{(j)}\}$ and $\{T_{kc_j}h^{(j)}\}$ of frames of translates such that for each j, c_j divides $c = \max_j c_j$. Consider now the pair

$$\left\{T_{kc}\left(T_{pc_j}g^{(j)}\right)\right\} \quad \text{and} \quad \left\{T_{kc}\left(T_{pc_j}h^{(j)}\right)\right\}$$

where p ranges from 0 to $\frac{c}{c_j} - 1$ for each j. Then this new system is really the same as our original system except we have now written it as a collection of vectors $\{T_{pc_j}g^{(j)}\}_{j,p}$ and $\{T_{pc_j}h^{(j)}\}_{j,p}$ with a constant translation parameter c. Considering that the length of the support of functions does not change via translation, we have therefore shown that dimension invariance applies to proportional non-uniform translation frames as well.

Corollary 3.1. *Let* $\{T_{kc_j}g^{(j)}\}$ *and* $\{T_{kc_j}h^{(j)}\}$ *be a dual pair of frame sequences of translates in* $\mathbb{F}^n$. *Suppose that for each* j, c_j *divides* $c = \max_j c_j$, *and that condition (2) holds. Then* $\{T_{kc_j}\tilde{g}^{(j)}\}$ *and* $\{T_{kc_j}\tilde{h}^{(j)}\}$ *(embedded in* $\mathbb{F}^{\tilde{n}}$*) remain a dual pair of frame sequences, provided each and every* c_j *divides* $\tilde{n}$.

Here, $\{x_n\}$ is called a *frame sequence* if $\{x_n\}$ is a frame in its own closed linear span $\overline{span}\{x_n\}$.

4. Multi-variable vector cases

We are to generalize Theorem 2.1 to multi-variable vector spaces (such as images). To this end, let G be an arbitrary abelian group and let $\mathcal{H} = \mathbb{F}^G$, *i.e.*, if $G = \bigoplus_{i=1}^m \mathbb{Z}_{n_i}$ then $\mathcal{H} = \bigotimes_{i=1}^m \mathbb{F}^{n_i}$. Take image analysis for instance, suppose $f(x,y)$ is a real valued image function of size $m \times n$. Then the abelian group $G = \mathbb{Z}_n \bigoplus \mathbb{Z}_m$, and $f \in \mathcal{H} \equiv \mathbb{R}^n \otimes \mathbb{R}^m$.

For $f \in \mathcal{H}$ we define the *support* of f as follows: $\text{supp}(f) = (a_1 \times \cdots \times a_m)$ when there is the smallest $a_1 \times \cdots \times a_m$ box where $f(x) = 0$ for all values of x outside of this box. In terms of the measure of the support box, we let $s_i = |a_i|$, and write

$$|\text{supp}(f)| = (s_1, ..., s_m).$$

For a collection $\{g^{(j)}\}$, we define

$$\max_j \left|\text{supp}\{g^{(j)}\}\right| = (S_1, ..., S_m), \quad S_i = \max_j \{s_i^{(j)}\}.$$

That is, $(S_1, ..., S_m)$ is the measure of the largest support of the collection $\{g^{(j)}\}$ in each variable among the collection index j.

Now let $c = (c_1, ..., c_m) \in G$ where c_i divides n_i for all $i = 1, ..., m$, and let

$$C = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_m \end{pmatrix}.$$

For each $i = 1, \cdots, m$, let $K_i = \{0, 1, \cdots, \frac{n_i}{c_i}\}$ and let $\mathcal{K} = K_1 \times \cdots \times K_m$. For $k \in \mathcal{K}$ define the translation operator $T_{Ck} : \mathcal{H} \to \mathcal{H}$ by $T_{Ck}f(x) = f(x - Ck)$. A frame of translates in this setting is a collection of the form $\{T_{Ck}g^{(j)}\}$, where $j \in J$ (the index set of the vectors $g^{(j)}$) and $k \in \mathcal{K}$.

Let $\mathcal{X} = \text{span}\{T_{Ck}g^{(j)}\}_{j\in J, k\in\mathcal{K}} \subseteq \mathcal{H}$, and let $\{T_{Ck}h^{(j)}\}$ be a dual frame satisfying

$$\forall f \in \mathcal{X}, \quad f = \sum_{j\in J}\sum_{k\in\mathcal{K}} \langle f, T_{Ck}g^{(j)}\rangle T_{Ck}h^{(j)}$$

Now let $\tilde{g}^{(j)}$ be a natural embedding of $g^{(j)}$ in $\tilde{\mathcal{H}} = \bigotimes_{i=1}^m \mathbb{F}^{\tilde{n}_i}$ (as before), and extending $\mathcal{X}$ to $\tilde{\mathcal{X}} = \text{span}\{T_{Ck}\tilde{g}^{(j)}\}_{j\in J, k\in\mathcal{K}} \subseteq \tilde{\mathcal{H}} = \bigotimes_{i=1}^m \mathbb{F}^{\tilde{n}_i}$ where each $\tilde{n}_i \geq n_i$, and each $\tilde{n}_i$ is a common multiple of the set $\{c_i\}_{i=1}^m$. The following similar dimension invariance property holds.

Theorem 4.1. *Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be defined as above. Suppose that $\{T_{Ck}g^{(j)}\}$ $\{T_{Ck}h^{(j)}\}$ are a dual pair of uniform frames of translates of $\mathcal{X}$, and that*

$$\max_j \left| supp\{g^{(j)}\}\right| + \max_j \left| supp\{h^{(j)}\}\right| \leq (n_1, ..., n_m),$$

then $\{T_{Ck}\tilde{g}^{(j)}\}$ and $\{T_{Ck}\tilde{h}^{(j)}\}$ are a dual pair of uniform frames of translates of $\tilde{\mathcal{X}}$.

Proof. Fix n_i unchanged for $i = 2, \cdots, m$, and assume that $\tilde{\mathcal{X}}_1 \subseteq \mathbb{F}^{\tilde{n}_1} \otimes (\bigotimes_{i=2}^m \mathbb{F}^{n_i})$. Then an argument identical to the proof of Theorem 2.1 over the first variable corresponding to $\tilde{n}_1$ establishes the result in $\tilde{\mathcal{X}}_1$. Extending $\tilde{\mathcal{X}}_1$ to $\tilde{\mathcal{X}}_i$ by extending one (and only one) variable of the very next index i for which $\tilde{n}_i > n_i$. The theorem now follows by induction on the index set $\{i : n'_i > n_i\}$ over the subspace $\tilde{\mathcal{X}}_i$, one at a time. □

We comment that non-uniform buy proportional translates in multi-variable cases can be treated similarly as in Section 3.

5. Gabor and multi-Gabor frames

Gabor frames. Let $\mathcal{H} = \mathbb{C}^n$, let M be a divisor of n, and let $b = \frac{1}{M}$. Define the *modulation operator* $E_b : \mathcal{H} \to \mathcal{H}$ by the formula

$$E_b g(x) = e^{2\pi i b x} g(x).$$

A frame for $\mathcal{H}$ of the form $\{E_{mb}T_{kc}g\}$ is called a *Gabor frame*.

Theorem 5.1. *Let* $\{E_{mb}T_{kc}g\}$ *and* $\{E_{mb}T_{kc}h\}$ *be a dual pair of Gabor frames for* $\mathbb{C}^n$ *and suppose* $|supp(g)|+|supp(h)| \leq n$*. Suppose* $\tilde{n} > n$ *and* $\tilde{n}$ *is a common multiple of* M *and* c*, then* $\{E_{mb}T_{kc}\tilde{g}\}$ *and* $\{E_{mb}T_{kc}\tilde{h}\}$ *is a dual pair of Gabor frames for* $\mathbb{C}^{\tilde{n}}$*.*

Proof. First observe that

$$E_bT_cg(x) = e^{2\pi ibx}g(x-c) = e^{2\pi ibc}T_cE_bg(x). \tag{4}$$

So for every $f \in \mathcal{H}$ we have

$$\begin{aligned} f &= \sum_{m=0}^{M-1}\sum_{k=0}^{N-1}\langle f, E_{mb}T_{kc}g\rangle E_{mb}T_{kc}h \\ &= \sum_{m=0}^{M-1}\sum_{k=0}^{N-1}\langle f, e^{2\pi ibmck}T_{kc}E_{mb}g\rangle e^{2\pi ibmck}T_{kc}E_{mb}h \\ &= \sum_{m=0}^{M-1}\sum_{k=0}^{N-1}\langle f, T_{kc}E_{mb}g\rangle T_{kc}E_{mb}h. \end{aligned}$$

Thus, $\{T_{kc}E_{mb}g\}$ and $\{T_{kc}E_{mb}h\}$ are a dual pair of frames of translates for $\mathcal{H}$ with $g^{(m)} = E_{mb}g$ and $h^{(m)} = E_{mb}h$. We also have that $\text{supp}(g^{(m)}) = \text{supp}(g)$ and $\text{supp}(h^{(m)}) = \text{supp}(h)$, so since we are assuming that $|\text{supp}(g)| + |\text{supp}(h)| \leq n$, the pair $\{T_{kc}g^{(m)}\}$ and $\{T_{kc}h^{(m)}\}$ satisfies (2). Therefore we can apply Theorem 2.1 to this pair to see that $\{T_{kc}\tilde{g}^{(m)}\}$ and $\{T_{kc}\tilde{h}^{(m)}\}$ are a dual pair of uniform frames of translates for $\mathbb{C}^{\tilde{n}}$. Now since M is also a divisor of $\tilde{n}$ we see that embedding satisfies $\tilde{g}^{(m)} = E_{mb}\tilde{g}$ and $\tilde{h}^{(m)} = E_{mb}\tilde{h}$, so by (4) the theorem is proved. □

Multi-Gabor expansions. Some applications call for multi-Gabor frames of the form $\{E_{mb_j}T_{kc_j}g^{(j)}\}$, e.g., [22], [23], etc. In this setting we have the following analog of our main result. We state it in the proportional nonuniform case.

Theorem 5.2. *Suppose that* $\{E_{mb_j}T_{kc_j}g^{(j)}\}$ *and* $\{E_{mb_j}T_{kc_j}h^{(j)}\}$ *are a dual pair of multi-Gabor frames for* $\mathbb{C}^n$ *where* n *is a common multiple of all integers in the set* $\{c_j, b_j\}_j$*. Suppose further that*

$$\max_j \left|supp(g^{(j)})\right| + \max_i \left|supp(h^{(i)})\right| \leq n.$$

If $\tilde{n} > n$ *is a common multiple of the set* $\{c_j, b_j\}$ *then* $\{E_{mb_j}T_{kc_j}\tilde{g}^{(j)}\}$ *and* $\{E_{mb_j}T_{kc_j}\tilde{h}^{(j)}\}$ *are a dual pair of multi-Gabor frames for* $\mathbb{C}^{\tilde{n}}$*.*

6. Construction of compactly supported duals

Given a uniform frame of translates $\{T_{kc}g^{(j)}\}$, the canonical dual $\{S^{-1}T_{kc}g^{(j)}\}$ is also a uniform frame of translates with the same translation parameter c and the same number of windows ℓ; however, it will almost always be the case that the pair $(\{T_{kc}g^{(j)}\}, \{S^{-1}T_{kc}g^{(j)}\})$ will not satisfy (2). Recall that any given frame $\{g_j\}$ may have infinitely many alternative duals. Our construction relies on the following characterization of alternative duals [21]. There are other constructions of compactly supported Gabor duals, e.g., [17].

Theorem 6.1. *Let $\mathcal{H} = \mathbb{F}^n$ and let $\{g_k\}_{k=1}^m$ be a frame for $\mathcal{H}$ with frame operator S. Then the set of all dual frames $\{h_j\}_{k=1}^m$ is given by*

$$h_k = S^{-1}g_k + \xi_k - \sum_{i=1}^{m} \langle S^{-1}g_k, g_i\rangle \xi_i, \tag{5}$$

where $\{\xi_j\} \subseteq \mathcal{H}$ is a (free) parametric sequence.

So if we start with a redundant frame of translates $\{T_{kc}g^{(j)}\}$ with compact support, and we want to find a compact dual $\{T_{kc}h^{(j)}\}$ (so that they satisfy (2)), all we have to do is to solve the system of linear equations given by:

$$h^{(j)}(x) = S^{-1}g^{(j)}(x) + \xi^{(j)}(x) - \sum \langle S^{-1}g^{(j)}, T_{kc}g^{(i)}\rangle T_{kc}\xi^{(i)}(x) = 0$$

for $x = 0, \cdots, M_1 - 1$ and $x = n - M_2, \cdots, n-1$. For redundant frames and for sufficiently small M_1 and M_2, solutions to $\{\xi^{(j)}\}$ of this system of equations always exists. Corresponding dual frame $\{h^{(j)}\}$ will be compact.

7. Essential support and approximate duals

In the last section we saw that there are in fact dual pairs of uniform frames of translates which satisfy (2) by the dual frame formula (5). We discuss in this section approximation scenarios where the condition (2) is satisfied only by δ-essential support. Here, let f_a be a truncation approximation of f with $f_a(x) = f(x) + \epsilon(x)$. The δ-essential support of f is defined by $\text{ess}_\delta\ \text{supp}(f) = \{\text{supp}(f_a) : \|\epsilon\| \le \delta\}$, $\delta > 0$.

Let $\mathcal{H} = \mathbb{F}^n$ and let $\{g_k\}$ be a frame of $\mathcal{X} \subseteq \mathcal{H}$ and suppose $\{h_k\}$ is a dual frame of $\{g_k\}$ in $\mathcal{X}$. Now consider the compact set $\{h_{k,a}\}$ with δ-essential support, or generally $h_{k,a} = h_k + \epsilon_k$. A set of this form is called an *approximate dual* of $\{g_k\}$. Note that in general

$$\sum_k \langle f, g_k\rangle h_{k,a} \ne f \ne \sum_k \langle f, h_{k,a}\rangle g_k. \tag{6}$$

In fact, $\{h_{k,a}\}$ may no longer be a sequence of the subspace $\mathcal{X}$ any more. Strictly speaking, summations in (6) become pseudoframe approximations of f.

For background on pseudoframes see [24]. Briefly, let $\mathcal{X} \subseteq \mathcal{H}$ be a subspace, and suppose $\{h_k\} \subseteq \mathcal{H}$ satisfies the frame inequalities for every f in $\mathcal{X}$. Then $\{h_k\}$ is a *pseudoframe* for $\mathcal{X}$ (note that we do not require $\{h_k\} \subseteq \mathcal{X}$). Similarly, if $\{g_k\}$ is a frame for $\mathcal{X}$ and $\{h_k\} \subseteq \mathcal{H}$ is a pseudoframe for $\mathcal{X}$ such that

$$f = \sum_k \langle f, h_k\rangle g_k$$

for every $f \in \mathcal{X}$ we call $\{h_k\}$ a *pseudo-dual frame* of $\{g_k\}$.

We would like to argue that $\{g_k\}$ and $\{h_{k,a}\}$ provides a good approximation of f through summations in (6), and the embedding of the pair $\{g_k\}$ and $\{h_{k,a}\}$ in a larger dimensional space would not cause error to accumulate, with conditions such as $\{\epsilon_k\}$ decaying sufficiently fast.

We begin with the following approximation presented in [25].

Theorem 7.1. *Let $\mathcal{X} \subseteq \mathcal{H}$ be (infinite dimensional) Hilbert spaces. Let $\{g_k\} \subseteq \mathcal{X}$ be a frame for $\mathcal{X}$, and $\{h_k\}$ be a (pseudo-)dual frame of $\{g_k\}$. Suppose $\{h_{k,a}\} \subseteq \mathcal{H}$ is an approximate pseudo-dual frame for $\{g_k\}$ where $h_{k,a} = h_k+\epsilon_k$, and $\|\epsilon_k\|_2 < \delta$. Suppose further that*

$$|\langle \epsilon_k, \epsilon_i\rangle| \leq \frac{\delta^2}{a^{|k-i|}}$$

for some $a > 1$. Then there is a $0 < C < \infty$ such that

$$\|f - \sum_k \langle f, h_{k,a}\rangle g_k\|_2 < C\delta\|f\|_2$$

for every $f \in \mathcal{X}$.

This leads to the following simple observation which we state as a proposition for later reference.

Proposition 7.2. *Let $\mathcal{X} \subseteq \mathcal{H} = \mathbb{F}^n$ and let $\{g_k\}$ be a (finite) frame for $\mathcal{X}$. Suppose $\{h_k\}$ is a dual (pseudo-)frame and $\{h_{k,a} = h_k + \epsilon_k\}$. Suppose further that $\max_k ||\epsilon_k||_p \leq \delta$ for some fixed $\delta > 0$ sufficiently small so that $\{h_{k,a}\}$ remains a Bessel sequence for $\mathcal{X}$. Then for each $1 \leq p \leq \infty$ there exists a positive constant C_p such that*

$$||f - \sum_k \langle f, h_{k,a}\rangle g_k||_p \leq C_p\delta||f||_p.$$

for every $f \in \mathcal{X}$.

Now suppose $(\{T_{kc}g^{(j)}\}, \{T_{kc}h^{(j)}\})$ are a dual pair of uniform frames of translates for $\mathcal{H} = \mathbb{F}^n$ that satisfy (2) by the measure of the δ-essential support, and let $h_a^{(j)}$ be such an approximate dual, i.e., $h_a^{(j)}(x) = h^{(j)}(x)$ for $x \in \text{ess}_\delta\ \text{supp}(h^{(j)})$, and write $h_a^{(j)} = h^{(j)} + \epsilon^{(j)}$. Suppose further that the stability as in Proposition 7.2 exists with $\{h_a^{(j)}\}$. Then the following proposition shows that if we embed this pair in a higher dimensional space then we do not accumulate any more error than we did in the original space.

Proposition 7.3. *Let $\mathcal{H} = \mathbb{F}^n$ and let $(\{T_{kc}g^{(j)}\}, \{T_{kc}h_a^{(j)}\})$ be an approximate dual pair of uniform frames of translates. Let $\tilde{\mathcal{H}} = \mathbb{F}^{\tilde{n}}$ where $\tilde{n}$ is divisible by c. Then*

$$||f - \sum_{j,k} \langle f, T_{kc}\tilde{h}_a^{(j)}\rangle T_{kc}\tilde{g}^{(j)}||_1 \leq C_1\delta||f||_1$$

and

$$||f - \sum_{j,k} \langle f, T_{kc}\tilde{g}^{(j)}\rangle T_{kc}\tilde{h}_a^{(j)}||_1 \leq C_1\delta||f||_1$$

for every $f \in \tilde{\mathcal{H}}$, where C_1 is the same constant from Proposition 7.2.

Proof. We will prove the first inequality, the proof of the second is analogous. Let $\{e_i\}_{i=0}^{n-1}$ be the standard orthonormal basis for $\mathcal{H}$ and $\{d_i\}_{i=0}^{n'-1}$ be the standard orthonormal basis for $\tilde{\mathcal{H}}$. Let

$$e_{i,a} = \sum_{j=1}^{\ell} \sum_k \langle e_i, T_{kc} h_a^{(j)} \rangle T_{kc} g^{(j)}.$$

Since the pair $(\{T_{kc}g^{(j)}\}, \{T_{kc}h_a^{(j)}\})$ satisfies (2) an argument similar to the proof of Theorem 2.1 shows that

$$d_{i,a} := \sum_{j=1}^{\ell} \sum_k \langle e_i, T_{kc} \tilde{h}_a^{(j)} \rangle T_{kc} \tilde{g}^{(j)} = \tilde{e}_{i,a},$$

where $d_{i,a}$ is a rename of $\tilde{e}_{i,a}$ which is the natural extension of $e_{i,a} \in \mathcal{H}$ in $\tilde{\mathcal{H}}$. Therefore, $||d_i - d_{i,a}||_1 \leq C_1\delta$ where C_1 is the same constant from *Theorem 7.2.* Now for any $f \in \tilde{\mathcal{H}}$ we have $f = \sum_{i=0}^{\tilde{n}-1} a_i d_i$ for some scalars $a_0, ..., a_{\tilde{n}-1}$, so

$$\begin{aligned} f_a : &= \sum_{j=1}^{\ell} \sum_k \langle f, T_{kc} \tilde{h}_a^{(j)} \rangle T_{kc} \tilde{g}^{(j)} \\ &= \sum_{j=1}^{\ell} \sum_k \langle \sum_{i=0}^{\tilde{n}-1} a_i d_i, T_{kc} \tilde{h}_a^{(j)} \rangle T_{kc} \tilde{g}^{(j)} \\ &= \sum_{i=0}^{\tilde{n}-1} a_i d_{i,a}. \end{aligned}$$

Thus,

$$\begin{aligned} ||f - f_a||_1 &= ||\sum_{i=0}^{\tilde{n}-1} a_i d_i - \sum_{i=0}^{\tilde{n}-1} a_i d_{i,a}||_1 \\ &= ||\sum_{i=0}^{\tilde{n}-1} a_i (d_i - d_{i,a})||_1 \\ &\leq \sum_{i=0}^{\tilde{n}-1} |a_i| ||d_i - d_{i,a}||_1 \\ &\leq C_1 \delta \sum_{i=0}^{\tilde{n}-1} |a_i| = C_1 \delta ||f||_1. \end{aligned}$$

□

Remark: Using a similar method we can obtain the same conclusion for the ∞-norm. That is, approximate duals through embedding in larger dimensional spaces does not accumulate approximation error than it does in the original space in the ∞-norm. We expect that an analogous statement should be true for any $1 \leq p \leq \infty$.

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