Homework 6

Math 471

1. Exercise 3.31, [DW], page 75.
2. Prove Theorem 3.42, [DW], page 80, parts (c), (d), (e).
3. Prove Theorem 3.43, [DW], page 80.
4. Prove the following theorem. Let $\phi \in L^2$, and $V_0 \equiv \{T_n\phi\}$. Define
$$V_j \equiv \left\{ f \in L^2 \,\middle|\, f(2^{-j}t) \in V_0 \right\}.$$
Then if $\{T_n\phi\}$ is an ONB of V_0, then $\{D_jT_n\phi\}$ is an ONB of V_j. Here $T_nf \equiv f(\cdot - n)$, and $D_jf = 2^{j/2}f(2^j\cdot)$.
5. [Optional] Let $\phi \in L^2(\mathbf{R})$ be such that $\hat{\phi}(\gamma) = 1$, for $|\gamma| \leq \Omega/2$, and $\hat{\phi}$ decays to zeros continuously for $|\gamma| > \Omega/2$. Derive a sampling formula of (or similar to) the following form:
$$\forall f \in PW_{\Omega/2}, \quad f(t) = \sum_{n\in\mathbf{Z}} f(\frac{n}{\Omega})\phi(t - \frac{n}{\Omega}),$$
where $PW_{\Omega/2}$ is the Paley Winer (band-limited) subspaces of functions $f \in L^2$ such that $\text{supp}(\hat{f}) \subseteq [-\Omega/2, \Omega/2]$. [Hint: work from the identity $\hat{f} = \hat{f}\hat{\phi}$ on $[-\Omega/2, \Omega/2]$, and follow the similar proof seen in [DW].]