Homework 5

1. Let ψ be given by the wavelet equation:

$$\psi(t) = \sqrt{2} \sum_k g_k \phi(2t-k).$$

Show that $\{\psi(t-k)\}_k$ is an ON set if and only if

$$|G(\gamma)|^2 + |G(\gamma+1/2)|^2 = 2,$$

where $G(\gamma)$ is the Fourier Transform of the sequence $\{g_k\}$.

2. Let $\{V_j, \phi\}$ be a given MRA. Let $\phi \in V_0$ and $\psi \in W_0$ be given by the scaling and wavelet equations, respectively,

$$\phi = \sum_k h_k \phi_{1k},$$

and

$$\psi = \sum_k g_k \phi_{1k}.$$

Suppose that

$$\phi_{1k}(t) = \sum_n a_{n,k} \phi(t-n) + \sum_n b_{n,k} \psi(t-n).$$

Show that

$$a_{n,k} = \overline{h_{k-2n}},$$

and

$$b_{n,k} = \overline{g_{k-2n}}.$$

3. Let $f \in L^2(\mathbf{R})$. Assume $f = \sum_j f_j$ where $f_i \perp f_j$ for all $i \neq j$. Show that $\|f\|^2 = \sum_j \|f_j\|^2$.

4. Let $\{V_j, \phi\}$ be a given MRA. Show that

$$\phi_{1k}(t) = \sum_n \overline{h_{k-2n}} \phi(t-n) + \sum_n \overline{g_{k-2n}} \psi(t-n) \qquad (1)$$

if and only if

$$\begin{aligned} |H(\gamma)|^2 + |G(\gamma)|^2 &= 2, \\ H(\gamma)\overline{G(\gamma)} + H(\gamma+\tfrac{1}{2})\overline{G(\gamma+\tfrac{1}{2})} &= 0. \end{aligned}$$

Hint: You would need to take the Fourier Transform of (1). You will see that there is a need to separate the equation into two by considering even k and odd k. You will also need to use

$$\Phi(\gamma) \equiv \sum_k |\hat{\phi}(\gamma+k)|^2 = 1.$$